

MULTIDEGREES OF TAME AUTOMORPHISMS OF  $\mathbb{C}^n$ 

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ABSTRACT. Let  $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a polynomial mapping. By the multidegree of  $F$  we mean  $\text{mdeg } F = (\deg F_1, \dots, \deg F_n) \in \mathbb{N}^n$ .

The aim of this paper is to study the following problem (especially for  $n = 3$ ): *for which sequence  $(d_1, \dots, d_n) \in \mathbb{N}^n$  is there a tame automorphism  $F$  of  $\mathbb{C}^n$  such that  $\text{mdeg } F = (d_1, \dots, d_n)$ ?* In other words we investigate the set  $\text{mdeg}(\text{Tame}(\mathbb{C}^n))$ , where  $\text{Tame}(\mathbb{C}^n)$  denotes the group of tame automorphisms of  $\mathbb{C}^n$ .

Since  $\text{mdeg}(\text{Tame}(\mathbb{C}^n))$  is invariant under permutations of coordinates, we may focus on the set  $\{(d_1, \dots, d_n) : d_1 \leq \dots \leq d_n\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^n))$ .

Obviously, we have  $\{(1, d_2, d_3) : 1 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)) = \{(1, d_2, d_3) : 1 \leq d_2 \leq d_3\}$ . Not obvious, but still easy to prove is the equality  $\{(2, d_2, d_3) : 2 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)) = \{(2, d_2, d_3) : 2 \leq d_2 \leq d_3\}$ .

In the paper, among other things, we give a complete description of the sets  $\{(3, d_2, d_3) : 3 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  and  $\{(5, d_2, d_3) : 5 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ . In the examination of the last set the most difficult part is to prove that  $(5, 6, 9) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ . To do this, we use the two dimensional Jacobian Conjecture (which is true for low degrees) and the Jung-van der Kulk Theorem.

As a surprising consequence of the method used in proving that  $(5, 6, 9) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ , we show that the existence of a tame automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (37, 70, 105)$  implies that the two dimensional Jacobian Conjecture is not true.

Also, we give a complete description of the following sets:  
 $\{(p_1, p_2, d_3) : 2 < p_1 < p_2 \leq d_3, p_1, p_2 \text{ prime numbers}\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ ,  
 $\{(d_1, d_2, d_3) : d_1 \leq d_2 \leq d_3, d_1, d_2 \in 2\mathbb{N} + 1, \gcd(d_1, d_2) = 1\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .  
 Using the description of the last set we show that  $\text{mdeg}(\text{Aut}(\mathbb{C}^3)) \setminus \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  is infinite.

We also obtain a (still incomplete) description of the set  $\{(4, d_2, d_3) : 4 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  and we give complete information about  $\text{mdeg } F^{-1}$  for  $F \in \text{Aut}(\mathbb{C}^2)$ .

## 0. INTRODUCTION

The object of principal interest in this paper is the multidegree (i.e. the sequence of the degrees of the coordinate functions) of a polynomial automorphism of the vector space  $\mathbb{C}^n$ . Let us mention that in the Scottish Book ([32], Problem 79) Mazur and Orlicz posed the following question: “If  $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a one-to-one polynomial map whose inverse is also a polynomial map, is each  $F_i$  of degree one?” In other words, they asked whether every polynomial automorphism of  $\mathbb{C}^n$  has multidegree  $(1, \dots, 1)$ . The answer to this question is obviously “no”, and in the Scottish Book itself one can find the following example: let  $1 \leq i \leq n$  and  $a = a(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n) \in \mathbb{C}[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$ . Then

$$E : \mathbb{C}^n \ni (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{i-1}, x_i + a, x_{i+1}, \dots, x_n) \in \mathbb{C}^n$$

is a polynomial automorphism with multidegree  $(1, \dots, 1, \deg a, 1, \dots, 1)$ . A map as above is called an *elementary* polynomial map. Taking finite compositions of such elementary maps and elements of the affine subgroup  $\text{Aff}(\mathbb{C}^n)$ , i.e. the group of polynomial automorphisms  $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that  $\deg F_i = 1$  for all  $i$ , we get automorphisms called *tame*.

In 1942 Jung [9] proved that each polynomial automorphism of  $k^2$ , where  $k$  is a field of characteristic zero, is tame. Later, in 1953, van der Kulk extended Jung's result to fields of arbitrary characteristic. Since then several authors have given other proofs of that result: Gutwirth [11] in 1961, Shafarevich [44] in 1966, Rentschler [40] in 1968, Makar-Limanov [31] in 1970, Nagata [34] in 1972, Abhyankar and Moh [1] in 1975, Dicks [6] in 1983, McKay and Wang [27] in 1988. The stronger statement, also called the Shafarevich-Nagata-Kombayashi theorem, saying that the group of all polynomial automorphisms of  $k^2$  is the amalgamated product of the affine subgroup and the subgroup of de Jonquières automorphisms over their intersection, can be found in [21], [16], [34], [6], [2] and without proof in [44].

From the result of Jung and van der Kulk it also follows that if  $(d_1, d_2)$  is the multidegree of an automorphism of  $\mathbb{C}^2$ , then  $d_1|d_2$  or  $d_2|d_1$  (see subsection 1.4).

Tame automorphisms are closely related to the problem of embedding of affine algebraic varieties. For example, in the proof of the famous Abhyankar-Moh-Suzuki theorem, saying that every embedding of a line in  $\mathbb{C}^2$  is rectifiable (i.e. a composition of the standard embedding  $\mathbb{C} \ni x \mapsto (x, 0) \in \mathbb{C}^2$  and an automorphism of  $\mathbb{C}^2$ ), tame automorphisms play a prominent role. This result, formulated in algebraic terms as follows: if  $f(T), g(T) \in k[T]$  and  $k[f(T), g(T)] = k[T]$ , then either  $\deg f(T) | \deg g(T)$  or  $\deg g(T) | \deg f(T)$ , was used by Segree [43] to “prove” the Jacobian Conjecture. The problem of embeddings of affine algebraic varieties was also considered by Jelonek [12, 13, 14], Kaliman [15], Srinivas [50] and Craighero [5].

Since Jung and van der Kulk proved their theorem, many authors have tried to prove or disprove the similar result for dimension  $n \geq 3$ , but without any results. The most famous candidate for a so-called wild automorphism (i.e. one that is not tame) was proposed by Nagata in 1972. It took more than thirty years to prove that the Nagata automorphism

$$\sigma : \mathbb{C}^3 \ni (x, y, z) \mapsto (x + 2y(y^2 + zx) - z(y^2 + zx)^2, y - z(y^2 + zx), z) \in \mathbb{C}^3$$

is indeed wild. This remarkable result was obtained by Shestakov and Umirbaev [47]. The two main ingredients in the proof of the above result are recalled as Theorems 8 and 12 (see subsections 2.1 and 2.3). These two theorems are also basic tools in our considerations concerning multidegrees of tame automorphisms of  $\mathbb{C}^3$ .

The paper is organized as follows. In section 1 we fix notation, recall basic definitions, and discuss the multidegree of polynomial automorphisms of  $\mathbb{C}^2$  (see subsection 1.4). The discussion is based on the Jung-van der Kulk result. In section 2 we recall the notion of a Poisson bracket of two polynomials, and two theorems due to Shestakov and Umirbaev (Theorems 8 and 12). They are the main tools used in the paper. We also prove that the degree of the Poisson bracket is an invariant of a linear change of coordinates (Lemma 10). This is a new result. In this section we also explain in detail that an example of a polynomial automorphism (Example 1) due to Shestakov and Umirbaev does not admit an elementary reduction, and recall

a theorem from number theory (Theorem 13) that will be useful in some parts of the paper.

In section 3 we collect some general results about multidegrees. Some of them were already published by the author: Proposition 14, Proposition 15 and Corollary 2 [17]. The other results in that section (except Theorem 26 due to Kuroda) are new. The most important results of that section are Proposition 15, Theorem 27 and Lemma 32.

In section 4 we discuss tame automorphisms of  $\mathbb{C}^3$  with multidegree of the form  $(p_1, p_2, d_3)$ ,  $2 < p_1 < p_2 \leq d_3$ , where  $p_1$  and  $p_2$  are prime numbers, and more generally, coprime odd numbers. In both cases we give a necessary and sufficient numerical condition on  $(p_1, p_2, d_3)$  to be the multidegree of tame automorphism of  $\mathbb{C}^3$ . The results of that section were already published by the author [18], and by the author and J. Zygałło [20].

Section 5 presents results due to the author [19]. They concern tame automorphisms with multidegree  $(3, d_2, d_3)$ ,  $3 \leq d_2 \leq d_3$ .

The results of sections 6 and 7 are new and concern tame automorphisms with multidegree  $(4, d_2, d_3)$ ,  $4 \leq d_2 \leq d_3$  (section 6), and  $(p, d_2, d_3)$ ,  $5 \leq p \leq d_2 \leq d_3$ , where  $p$  is a prime (section 7). It is of interest that in showing that there is no tame automorphism of  $\mathbb{C}^3$  with multidegree  $(5, 6, 9)$ , we use the Jacobian Conjecture (actually the Moh theorem). On the other hand, it is very surprising that the existence of a tame automorphism of  $\mathbb{C}^3$  with multidegree  $(37, 70, 105)$  implies that the two-dimensional Jacobian Conjecture is false (this is proved in section 7).

In section 8 we present a result due to J. Zygałło [52], and in the last section we give new results on the multidegree of the inverse of a polynomial automorphism of  $\mathbb{C}^2$ .

## 1. NOTATIONS, BASIC DEFINITIONS AND TWO-DIMENSIONAL CASE

**1.1. Notation.** We assume that  $0 \in \mathbb{N}$ , and we denote by  $\mathbb{N}^*, \mathbb{Z}^*, \mathbb{C}^*$ , respectively,  $\mathbb{N} \setminus \{0\}, \mathbb{Z} \setminus \{0\}, \mathbb{C} \setminus \{0\}$ . By  $\mathbb{C}[X_1, \dots, X_n]$  we denote the polynomial ring in  $n$  variables over  $\mathbb{C}$ . In particular,  $X_1, \dots, X_n$  denote variables, and  $x_1, \dots, x_n$  denote coordinates in  $\mathbb{C}^n$ . We will work over the complex field  $\mathbb{C}$ , but all results remain valid over any algebraically closed field of characteristic zero.

For any  $f \in \mathbb{C}[X_1, \dots, X_n]$ ,  $\deg f$  denotes the usual total degree of  $f$ . We say that  $f$  is homogeneous if  $f$  is a sum of monomials of the same degree. We denote by  $\bar{f}$  the leading form of  $f$ , i.e. the homogeneous part of  $f$  of the maximal degree. Of course,  $\deg f = \deg \bar{f}$ .

Moreover,  $\gcd(d_1, \dots, d_n)$  and  $\text{lcm}(d_1, \dots, d_n)$  denote the greatest common divisor of  $d_1, \dots, d_n$  and least common multiply of  $d_1, \dots, d_n$ , respectively.

**1.2. Examples of polynomial automorphisms.** First of all, recall that a *polynomial mapping*  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a mapping whose coordinate functions  $F_i$ , where  $F = (F_1, \dots, F_n)$ , are polynomials. By a *polynomial automorphism* of  $\mathbb{C}^n$  (later, just *automorphism*) we mean a polynomial mapping  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that there exists a polynomial mapping  $G : \mathbb{C}^n \rightarrow \mathbb{C}^n$  with  $F \circ G = G \circ F = \text{id}_{\mathbb{C}^n}$ . We then also say that  $F$  is invertible. The group of all polynomial automorphisms of  $\mathbb{C}^n$  is denoted by  $\text{Aut}(\mathbb{C}^n)$ .

Polynomial automorphisms play a prominent role in affine algebraic geometry [32, 45]. Typical problems are the Jacobian Problem [3, 4, 9, 21, 34, 35, 36, 37, 38],

existence of wild automorphisms [8, 47, 48, 49], the inverse formula [26, 27, 28, 29] or stable tameness [46].

There are some special kinds of polynomial automorphisms of  $\mathbb{C}^n$ :

- Affine polynomial automorphisms, i.e. polynomial automorphisms  $F = (F_1, \dots, F_n)$  such that  $\deg F_i = 1$  for  $i = 1, \dots, n$ . The set of all such automorphisms will be denoted  $\text{Aff}(\mathbb{C}^n)$ ; it is a subgroup of  $\text{Aut}(\mathbb{C}^n)$ .
- Linear automorphisms, i.e. affine automorphisms  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that  $F(0, \dots, 0) = (0, \dots, 0)$ .

This is of course the same as the general linear group, denoted  $GL_n(\mathbb{C})$ .

- Elementary automorphisms, i.e. maps of the form

$$F : \mathbb{C}^n \ni (x_1, \dots, x_n) \mapsto (x_1, \dots, x_i + f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \dots, x_n) \in \mathbb{C}^n$$

for some  $i \in \{1, \dots, n\}$  and  $f \in \mathbb{C}[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$ .

One can easily see that

$$F^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_i - f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \dots, x_n).$$

- Triangular automorphisms, i.e. maps of the form

$$(1) \quad F : \mathbb{C}^n \ni (x_1, \dots, x_n) \mapsto (x_1, x_2 + f_1(x_1), \dots, x_n + f_{n-1}(x_1, \dots, x_{n-1})) \in \mathbb{C}^n,$$

where  $f_1 \in \mathbb{C}[X_1]$ ,  $f_2 \in \mathbb{C}[X_1, X_2]$ ,  $\dots$ ,  $f_{n-1} \in \mathbb{C}[X_1, \dots, X_{n-1}]$ .

One can check that  $F$  is invertible and

$$F^{-1} \left( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \right) = \begin{pmatrix} x_1 \\ x_2 - f_1(x_1) \\ x_3 - f_2(x_1, x_2 - f_1(x_1)) \\ \vdots \end{pmatrix}.$$

We will also say that  $F$  is triangular if  $F$  is of the form (1) after some permutation of variables.

- De Jonquière's automorphisms, i.e. mappings of the form

$$(2) \quad F : \mathbb{C}^n \ni \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} a_1 x_1 + f_1(x_2, \dots, x_n) \\ a_2 x_2 + f_2(x_3, \dots, x_n) \\ \vdots \\ a_n x_n + f_n \end{pmatrix} \in \mathbb{C}^n,$$

where  $a_i \in \mathbb{C}^*$ ,  $f_i \in \mathbb{C}[X_{i+1}, \dots, X_n]$  for all  $1 \leq i \leq n-1$  and  $f_n \in \mathbb{C}$ . We then write  $F \in J(\mathbb{C}^n)$ .

As for triangular mappings, one can check that if  $F \in J(\mathbb{C}^n)$ , then  $F$  is invertible. Also, one can verify that  $J(\mathbb{C}^n)$  is a subgroup of  $\text{Aut}(\mathbb{C}^n)$ .

- Tame automorphisms, i.e. compositions of a finite number of affine and triangular automorphisms. Sometimes a tame automorphism is defined as a composition of a finite number of affine and elementary automorphisms, or as a composition of a finite number of affine and de Jonquière's automorphisms. One can check that all these definitions are equivalent.

To end this section, recall that for any polynomial mapping  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  we have the  $\mathbb{C}$ -homomorphism  $F^* : \mathbb{C}[X_1, \dots, X_n] \rightarrow \mathbb{C}[X_1, \dots, X_n]$  defined by

$$F^* : \mathbb{C}[X_1, \dots, X_n] \ni h \mapsto h \circ F \in \mathbb{C}[X_1, \dots, X_n],$$

and for any  $\mathbb{C}$ -homomorphism  $\Phi : \mathbb{C}[X_1, \dots, X_n] \rightarrow \mathbb{C}[X_1, \dots, X_n]$  we have the polynomial mapping  $\Phi_* : \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined as

$$\Phi_* : \mathbb{C}^n \ni (x_1, \dots, x_n) \mapsto (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n)) \in \mathbb{C}^n,$$

where  $F_i = \Phi(X_i)$ . Moreover, recall that  $(F^*)_* = F$ ,  $(\Phi_*)^* = \Phi$ , and  $F$  is an automorphism if and only if  $F^*$  is a  $\mathbb{C}$ -automorphism of  $\mathbb{C}[X_1, \dots, X_n]$ . Thus one can translate the notions of affine, linear, elementary, triangular and tame automorphisms of  $\mathbb{C}^n$  into the language of  $\mathbb{C}$ -automorphisms of  $\mathbb{C}[X_1, \dots, X_n]$ .

**1.3. Degree, bidegree and multidegree.** Let  $F = (F_1, \dots, F_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be any polynomial map. By the *degree* of  $F$ , denoted  $\deg F$ , we mean the number

$$\deg F = \max \{ \deg F_1, \dots, \deg F_n \},$$

and by the *multidegree* of  $F$ , denoted  $\text{mdeg } F$ , we mean the sequence of natural numbers

$$\text{mdeg } F = (\deg F_1, \dots, \deg F_n).$$

For  $n = 2$  the multidegree is called bidegree, and denoted  $\text{bideg}$ . (see e.g. [7]).

For a fixed  $n \in \mathbb{N}$ , we will also consider the mappings

$$\deg : \text{End}(\mathbb{C}^n) \ni F \mapsto \deg F \in \mathbb{N}$$

and

$$\text{mdeg} : \text{End}(\mathbb{C}^n) \ni F \mapsto \text{mdeg } F \in \mathbb{N}^n,$$

where  $\text{End}(\mathbb{C}^n)$  denotes the set of all polynomial mappings  $\mathbb{C}^n \rightarrow \mathbb{C}^n$ .

One of the main goals of this paper is to obtain a description of the sets

$$\text{mdeg}(\text{Aut}(\mathbb{C}^n)), \text{mdeg}(\text{Tame}(\mathbb{C}^n)) \subset \mathbb{N}^n.$$

If  $n = 1$  the answer is

$$\text{mdeg}(\text{Aut}(\mathbb{C}^1)) = \text{mdeg}(\text{Tame}(\mathbb{C}^1)) = \{1\}.$$

The description for  $n = 2$ , based on a theorem of Jung and van der Kulk, will be given in the next subsection. The answer for  $n \geq 3$  is much more complicated, and will be investigated in the rest of the paper. The very first result in this direction says that  $(3, 4, 5) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  [17]. The next results obtained by the author [18, 19, 20] are also included.

Since for any  $(F_1, \dots, F_n) \in \text{Aut}(\mathbb{C}^n)$  we have  $\deg F_i \geq 1$ ,  $i = 1, \dots, n$ , and since for any permutation  $\sigma$  of  $\{1, \dots, n\}$  and any sequence  $(d_1, \dots, d_n) \in \mathbb{N}^n$  we have

$$(d_1, \dots, d_n) \in \text{mdeg}(\text{Tame}(\mathbb{C}^n)) \iff (d_{\sigma(1)}, \dots, d_{\sigma(n)}) \in \text{mdeg}(\text{Tame}(\mathbb{C}^n))$$

and

$$(d_1, \dots, d_n) \in \text{mdeg}(\text{Aut}(\mathbb{C}^n)) \iff (d_{\sigma(1)}, \dots, d_{\sigma(n)}) \in \text{mdeg}(\text{Aut}(\mathbb{C}^n)),$$

in our considerations we can always assume that  $1 \leq d_1 \leq \dots \leq d_n$ . In other words, we will consider the sets

$$\text{mdeg}(\text{Tame}(\mathbb{C}^n)) \cap \{(d_1, \dots, d_n) : 1 \leq d_1 \leq \dots \leq d_n\} \subset \mathbb{N}^n$$

and

$$\text{mdeg}(\text{Aut}(\mathbb{C}^n)) \cap \{(d_1, \dots, d_n) : 1 \leq d_1 \leq \dots \leq d_n\} \subset \mathbb{N}^n.$$

**1.4. Jung and van der Kulk result.** Before giving a description of the set  $\text{mdeg}(\text{Tame}(\mathbb{C}^2))$ , we recall the following two classical results.

**Proposition 1.** ([7], Corollary 5.1.3)  *$\text{Tame}(\mathbb{C}^2)$  is the amalgamated product of  $\text{Aff}(\mathbb{C}^2)$  and  $J(\mathbb{C}^2)$  over their intersection, i.e.  $\text{Tame}(\mathbb{C}^2)$  is generated by these two groups and if  $\tau_i \in J(\mathbb{C}^2) \setminus \text{Aff}(\mathbb{C}^2)$  and  $\lambda_i \in \text{Aff}(\mathbb{C}^2) \setminus J(\mathbb{C}^2)$ , then  $\tau_1 \circ \lambda_1 \circ \dots \circ \tau_n \circ \lambda_n \circ \tau_{n+1}$  does not belong to  $\text{Aff}(\mathbb{C}^2)$ .*

Let us here recall the definition of an amalgamated product, following [42].

**Definition 1.** *Let  $G$  be a group and let  $A, B$  be two subgroups with  $C = A \cap B$ . We denote by  $\Phi$  (resp.  $\Psi$ ) a complete set of representatives of the left coset space  $A/C$  (resp.  $B/C$ ) subject only to the restriction that the representative of  $C$  itself is the neutral element of  $G$ . We say that  $G$  is an amalgamated product of  $A$  and  $B$  over  $C$  if every element  $g \in G$  can be written uniquely as  $g = \varphi_0 \psi_1 \varphi_1 \psi_2 \dots \varphi_{n-1} \psi_n \varphi_n \gamma$  for suitable  $n \in \mathbb{N}$ ,  $\varphi_0, \dots, \varphi_n \in \Phi$ ,  $\psi_1, \dots, \psi_n \in \Psi$ ,  $\gamma \in C$ , where only  $\varphi_0, \varphi_n$  and  $\gamma$  may be the neutral element.*

The second result is the following

**Corollary 2.** ([7], Corollary 5.1.6) *Let  $F = (F_1, F_2) \in \text{Tame}(\mathbb{C}^2)$  with  $\text{bideg } F = (d_1, d_2)$ . Let  $h_i$  denote the homogeneous component of  $F_i$  of degree  $d_i$ . Then:*

- a)  $d_1 | d_2$  or  $d_2 | d_1$ .
- b) If  $\deg F > 1$ , then we have:
  - i) if  $d_1 < d_2$ , then  $h_2 = ch_1^{\frac{d_2}{d_1}}$  for some  $c \in \mathbb{C}$ ,
  - ii) if  $d_2 < d_1$ , then  $h_1 = ch_2^{\frac{d_1}{d_2}}$  for some  $c \in \mathbb{C}$ ,
  - iii) if  $d_1 = d_2$ , then there exists  $\lambda \in \text{Aff}(\mathbb{C}^2)$  such that  $\deg \tilde{F}_1 > \deg \tilde{F}_2$ , where  $\tilde{F} = (\tilde{F}_1, \tilde{F}_2) = \lambda \circ F$ .

From the above corollary we obtain

$$\text{mdeg}(\text{Tame}(\mathbb{C}^2)) \cap \{(d_1, d_2) : 1 \leq d_1 \leq d_2\} \subset \{(d_1, d_2) \in (\mathbb{N}^*)^2 : d_1 | d_2\}.$$

Since, for  $d_1 | d_2$ , and

$$\begin{aligned} F_1 & : \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{d_1}, y) \in \mathbb{C}^2, \\ F_2 & : \mathbb{C}^2 \ni (u, v) \mapsto \left(u, v + u^{\frac{d_2}{d_1}}\right) \in \mathbb{C}^2, \end{aligned}$$

$F_2 \circ F_1$  is a tame automorphism of  $\mathbb{C}^2$  with  $\text{mdeg}(F_2 \circ F_1) = (d_1, d_2)$ , we see that

$$\text{mdeg}(\text{Tame}(\mathbb{C}^2)) \cap \{(d_1, d_2) : 1 \leq d_1 \leq d_2\} = \{(d_1, d_2) \in (\mathbb{N}^*)^2 : d_1 | d_2\}.$$

To obtain a description of the set  $\text{mdeg}(\text{Aut}(\mathbb{C}^2))$ , we also need the following result due to Jung [9] and van der Kulk [21].

**Theorem 3.** (Jung-van der Kulk, see e.g. [7], Theorem 5.1.11) *We have  $\text{Aut}(\mathbb{C}^2) = \text{Tame}(\mathbb{C}^2)$ . More precisely,  $\text{Aut}(\mathbb{C}^2)$  is the amalgamated product of  $\text{Aff}(\mathbb{C}^2)$  and  $J(\mathbb{C}^2)$  over their intersection.*

Using Theorem 3, we of course obtain

$$\text{mdeg}(\text{Aut}(\mathbb{C}^2)) = \text{mdeg}(\text{Tame}(\mathbb{C}^2)),$$

and so

$$\text{mdeg}(\text{Aut}(\mathbb{C}^2)) \cap \{(d_1, d_2) : 1 \leq d_1 \leq d_2\} = \{(d_1, d_2) \in (\mathbb{N}^*)^2 : d_1 | d_2\}.$$

A crucial result, used in the proof of the Jung-van der Kulk result, is the following lemma and the notion of elementary reduction.

**Lemma 4.** (see e.g. [7], Lemma 10.2.4) *Let  $f, g \in \mathbb{C}[X, Y]$  be homogeneous polynomials such that  $\text{Jac}(f, g) = 0$ . Then there exists a homogeneous polynomial  $h$  such that:*

- i)  $f = c_1 h^{n_1}$  and  $g = c_2 h^{n_2}$  for some integers  $n_1, n_2 \geq 0$  and  $c_1, c_2 \in \mathbb{C}^*$ .
- ii)  $h$  is not of the form  $ch_0^s$  for any  $c \in k^*$ , any  $h_0 \in k[x, y]$  and any integer  $s > 1$ .

Recall that an automorphism  $F = (F_1, \dots, F_n)$  admits an *elementary reduction* if there exists an elementary automorphism  $\tau : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that for  $G = (G_1, \dots, G_n) = \tau \circ F$  we have

$$\text{mdeg } G < \text{mdeg } F,$$

i.e.

$$\deg G_i \leq \deg F_i \quad \text{for all } i = 1, \dots, n$$

and

$$\deg G_i < \deg F_i \quad \text{for some } i \in \{1, \dots, n\}.$$

We then say that  $G$  is an elementary reduction of  $F$ . One can easily notice that  $F$  admits an elementary reduction if there exists  $i \in \{1, \dots, n\}$  and a polynomial  $g \in \mathbb{C}[Y_1, \dots, Y_{n-1}]$  such that

$$\deg(F_i - g(F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_n)) < \deg F_i.$$

We will also need the following generalization of the above lemma.

**Proposition 5.** *Let  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  be homogeneous, algebraically dependent polynomials. Then there exists a homogeneous polynomial  $h \in \mathbb{C}[X_1, \dots, X_n]$  such that:*

- i)  $f = c_1 h^{n_1}$  and  $g = c_2 h^{n_2}$  for some integers  $n_1, n_2 \geq 0$  and  $c_1, c_2 \in \mathbb{C}^*$ .
- ii)  $h$  is not of the form  $ch_0^s$  for any  $c \in \mathbb{C}^*$ , any  $h_0 \in \mathbb{C}[X_1, \dots, X_n]$  and any integer  $s > 1$ .

One can obtain the above result using Lemma 2 in [51].

## 2. MAIN TOOLS

**2.1. Poisson bracket and degree of polynomials.** In this section we present the first main tool which we will use in our considerations: the Poisson bracket of two polynomials and a theorem that estimates from below the degree of a polynomial of the form  $h(f, g)$ , where  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  and  $h \in \mathbb{C}[X, Y]$ .

We start with the definition of a  $*$ -reduced pair.

**Definition 2.** ([47], Definition 1) *A pair  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  is called  $*$ -reduced if*

- (i)  $f, g$  are algebraically independent;
- (ii)  $\bar{f}, \bar{g}$  are algebraically dependent;
- (iii)  $\bar{f} \notin \mathbb{C}[\bar{g}]$  and  $\bar{g} \notin \mathbb{C}[\bar{f}]$ .

Moreover, we say that  $f, g$  is a  $p$ -reduced pair if  $f, g$  is a  $*$ -reduced pair with  $\deg f < \deg g$  and  $p = \frac{\deg f}{\gcd(\deg f, \deg g)}$ .

One may ask whether  $p$  can be equal to 1 for a  $p$ -reduced pair  $f, g$ . The answer is given by the following

**Proposition 6.** *If  $f, g$  is a  $p$ -reduced pair, then  $p > 1$ .*

*Proof.* If  $f, g$  is  $p$ -reduced, then  $\bar{f}$  and  $\bar{g}$  are algebraically dependent. This means, by Proposition 5, that there is a homogeneous polynomial  $h$  such that

$$\bar{f} = \alpha h^l \quad \text{and} \quad \bar{g} = \beta h^m$$

for some  $\alpha, \beta \in \mathbb{C}^*$  and  $l, m \in \mathbb{N}$ . Assume that  $p = \frac{\deg f}{\gcd(\deg f, \deg g)} = 1$ . Then  $l|m$ , and so  $\bar{g} = \gamma \bar{f}^r$  for  $r = \frac{m}{l}$  and  $\gamma \in \mathbb{C}^*$ . This contradicts condition (iii) of Definition 2.  $\square$

For any  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  we denote by  $[f, g]$  the Poisson bracket of  $f$  and  $g$ , i.e. the following formal sum:

$$\sum_{1 \leq i < j \leq n} \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j],$$

where  $[X_i, X_j]$  are formal objects satisfying the condition

$$[X_i, X_j] = -[X_j, X_i] \quad \text{for all } i, j.$$

We also define

$$\deg [X_i, X_j] = 2 \quad \text{for all } i \neq j,$$

$\deg 0 = -\infty$  and

$$\deg [f, g] = \max_{1 \leq i < j \leq n} \deg \left\{ \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j] \right\}.$$

Since  $2 - \infty = -\infty$ , we have

$$\deg [f, g] = 2 + \max_{1 \leq i < j \leq n} \deg \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right),$$

and hence

$$(3) \quad \deg [f, g] \leq \deg f + \deg g.$$

Another inequality involving the degree of a Poisson bracket will be a consequence of Proposition 7 below, in which  $\frac{\partial(F_1, \dots, F_r)}{\partial(X_1, \dots, X_n)}$  means the Jacobian matrix (not necessarily quadratic) of the mapping  $(F_1, \dots, F_r) : \mathbb{C}^n \rightarrow \mathbb{C}^r$ .

**Proposition 7.** *If  $F_1, \dots, F_r \in \mathbb{C}[X_1, \dots, X_n]$ , then*

$$\text{rank} \frac{\partial(F_1, \dots, F_r)}{\partial(X_1, \dots, X_n)} = \text{trdeg}_{\mathbb{C}} \mathbb{C}(F_1, \dots, F_r).$$

One can deduce the above result from [25, Chap. X, Prop. 10]. The version for  $r = n$  can also be found in [7, Prop. 1.2.9].

By Proposition 7 and the definition of the degree of a Poisson bracket we obtain the following remark.

**Remark 1.**  *$f, g \in \mathbb{C}[X_1, \dots, X_n]$  are algebraically independent if and only if  $\deg [f, g] \geq 2$ .*

We also have the following



**Remark 2.** For any  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  the following conditions are equivalent:

- (1)  $\deg[f, g] = \deg f + \deg g$ ,
- (2)  $\bar{f}, \bar{g}$  are algebraically independent.

*Proof.* Let

$$f = f_0 + \dots + f_d, \quad g = g_0 + \dots + g_m$$

be the homogeneous decompositions of  $f$  and  $g$ . Since

$$[f, g] = \sum_{i,j} [f_i, g_j] = [f_d, g_m] + \sum_{i < d \text{ or } j < m} [f_i, g_j]$$

and

$$\deg[f_i, g_j] \leq \deg f_i + \deg g_j = i + j < d + m,$$

for  $i < d$  or  $j < m$ , it follows that

$$\deg[f, g] = d + m \iff \deg[f_d, g_m] = d + m.$$

But, since  $f_d$  and  $g_m$  are homogeneous polynomials of degrees  $d$  and  $m$ , respectively, by the definition of Poisson bracket we have

$$\deg[f_d, g_m] = d + m \iff [f_d, g_m] \neq 0.$$

The last condition, by Proposition 7, is equivalent to  $f_d, g_m$  being algebraically independent.  $\square$

Recall the following theorem due to Shestakov and Umirbaev.

**Theorem 8.** ([47], Theorem 2) Let  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  be a  $p$ -reduced pair, and let  $G(X, Y) \in k[X, Y]$  with  $\deg_Y G(X, Y) = pq + r, 0 \leq r < p$ . Then

$$\deg G(f, g) \geq q(p \deg g - \deg g - \deg f + \deg[f, g]) + r \deg g.$$

Notice that the estimate from Theorem 8 is true even if the condition (ii) of Definition 2 is not satisfied. Indeed, if  $G = \sum_{i,j} a_{i,j} X^i Y^j$ , we then have, by the algebraic independence of  $\bar{f}$  and  $\bar{g}$ ,

$$\begin{aligned} \deg G(f, g) &= \max_{i,j} \deg(a_{i,j} f^i g^j) \geq \deg_Y G(X, Y) \cdot \deg g \\ &= (qp + r) \deg g \geq q(p \deg g - \deg f - \deg g + \deg[f, g]) + r \deg g. \end{aligned}$$

The last inequality is a consequence of the fact that  $\deg[f, g] \leq \deg f + \deg g$ .

Notice that the above calculations are also valid for  $p = 1$  (when the pair  $f, g$  does not satisfy the condition (ii) of Definition 2,  $p$  may be equal to one).

Thus we have the following proposition.

**Proposition 9.** Let  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  satisfy conditions (i) and (iii) of Definition 2. Assume that  $\deg f < \deg g$ , put

$$p = \frac{\deg f}{\gcd(\deg f, \deg g)},$$

and let  $G(X, Y) \in \mathbb{C}[X, Y]$  with  $\deg_Y G(X, Y) = pq + r, 0 \leq r < p$ . Then

$$\deg G(f, g) \geq q(p \deg g - \deg g - \deg f + \deg[f, g]) + r \deg g.$$

**2.2. Degree of a Poisson bracket and a linear change of coordinates.** This section is devoted to showing the following lemma saying that the degree of a Poisson bracket is invariant under a linear change of coordinates.

**Lemma 10.** *If  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  and  $L \in GL_n(\mathbb{C})$ , then*

$$\deg[L^*(f), L^*(g)] = \deg[f, g],$$

where  $L^*(h) = h \circ L$  for any  $h \in \mathbb{C}[X_1, \dots, X_n]$ .

We first show

**Proposition 11.** *If  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  and  $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is any linear map, then*

$$\deg[L^*(f), L^*(g)] \leq \deg[f, g].$$

*Proof.* It is easy to see that for every  $h \in \mathbb{C}[X_1, \dots, X_n]$  we have (here we allow  $L^*(h_d) = 0$  even if  $h_d \neq 0$ )

$$[L^*(h)]_d = L^*(h_d),$$

where the subscript  $d$  denotes the homogeneous part of degree  $d$ . We also have

$$[Jac^{ij}(f, g)]_d = \sum_{k+l=d+2} Jac^{ij}(f_k, g_l),$$

where

$$Jac^{ij}(f, g) = Jac^{X_i X_j}(f, g) = \det \begin{bmatrix} \frac{\partial f}{\partial X_i} & \frac{\partial f}{\partial X_j} \\ \frac{\partial g}{\partial X_i} & \frac{\partial g}{\partial X_j} \end{bmatrix}.$$

By the above equalities we have

$$\begin{aligned} (4) \quad [Jac^{ij}(L^*(f), L^*(g))]_d &= \sum_{k+l=d+2} Jac^{ij}(L^*(f)_k, L^*(g)_l) \\ &= \sum_{k+l=d+2} Jac^{ij}(L^*(f_k), L^*(g_l)). \end{aligned}$$

Since for any  $h \in \mathbb{C}[X_1, \dots, X_n]$  and  $r \in \{1, \dots, n\}$  we have

$$\frac{\partial L^*(h)}{\partial X_r} = \frac{\partial(h \circ L)}{\partial X_r} = \sum_{s=1}^n \frac{\partial h}{\partial X_s}(L) \cdot a_{sr},$$

where  $(a_{ij})$  is the matrix of the mapping  $L$ , it follows that

$$\begin{aligned}
 (5) \quad Jac^{ij}(L^*(f_k), L^*(g_l)) &= \det \begin{bmatrix} \sum_{r=1}^n \frac{\partial f_k}{\partial X_r}(L) \cdot a_{ri} & \sum_{r=1}^n \frac{\partial f_k}{\partial X_r}(L) \cdot a_{rj} \\ \sum_{s=1}^n \frac{\partial g_l}{\partial X_s}(L) \cdot a_{si} & \sum_{s=1}^n \frac{\partial g_l}{\partial X_s}(L) \cdot a_{sj} \end{bmatrix} \\
 &= \sum_{r,s=1}^n \frac{\partial f_k}{\partial X_r}(L) \cdot a_{ri} \cdot \frac{\partial g_l}{\partial X_s}(L) \cdot a_{sj} - \sum_{r,s=1}^n \frac{\partial f_k}{\partial X_r}(L) \cdot a_{rj} \cdot \frac{\partial g_l}{\partial X_s}(L) \cdot a_{si} \\
 &= \sum_{r,s=1}^n \left[ \frac{\partial f_k}{\partial X_r}(L) \cdot a_{ri} \cdot \frac{\partial g_l}{\partial X_s}(L) \cdot a_{sj} - \frac{\partial f_k}{\partial X_s}(L) \cdot a_{sj} \cdot \frac{\partial g_l}{\partial X_r}(L) \cdot a_{ri} \right] \\
 &= \sum_{r,s=1}^n Jac^{rs}(f_k, g_l)(L) \cdot a_{ri} a_{sj} \\
 &= \sum_{1 \leq r < s \leq n} Jac^{rs}(f_k, g_l)(L) \cdot a_{ri} a_{sj} + \sum_{1 \leq s < r \leq n} Jac^{rs}(f_k, g_l)(L) \cdot a_{ri} a_{sj} \\
 &= \sum_{1 \leq r < s \leq n} Jac^{rs}(f_k, g_l)(L) \cdot a_{ri} a_{sj} - \sum_{1 \leq r < s \leq n} Jac^{rs}(f_k, g_l)(L) \cdot a_{si} a_{rj} \\
 &= \sum_{1 \leq r < s \leq n} Jac^{rs}(f_k, g_l)(L) \det \begin{bmatrix} a_{ri} & a_{rj} \\ a_{si} & a_{sj} \end{bmatrix}.
 \end{aligned}$$

Now, by (4) and (5), we have

$$\begin{aligned}
 (6) \quad & [Jac^{ij}(L^*(f), L^*(g))]_d \\
 &= \sum_{k+l=d+2} \sum_{1 \leq r < s \leq n} Jac^{rs}(f_k, g_l)(L) \det \begin{bmatrix} a_{ri} & a_{rj} \\ a_{si} & a_{sj} \end{bmatrix} \\
 &= \sum_{1 \leq r < s \leq n} \left( \sum_{k+l=d+2} Jac^{rs}(f_k, g_l) \right) (L) \det \begin{bmatrix} a_{ri} & a_{rj} \\ a_{si} & a_{sj} \end{bmatrix}.
 \end{aligned}$$

Take any  $d > \deg[f, g]$ . Then

$$(7) \quad \sum_{k+l=d+2} Jac^{rs}(f_k, g_l) = 0$$

for all pairs  $r, s$  satisfying  $1 \leq r < s \leq n$ . Thus, by (6) and (7), we obtain

$$(8) \quad [Jac^{ij}(L^*(f), L^*(g))]_d = 0$$

for all  $i, j$ . The above equalities (for all  $i, j$ ) mean that  $\deg[L^*(f), L^*(g)] < d$ . Since we can take  $d = \deg[f, g] + 1, \deg[f, g] + 2, \dots$  we obtain

$$(9) \quad \deg[L^*(f), L^*(g)] \leq \deg[f, g].$$

□

Now, we can prove Lemma 10.

*Proof.* By the above proposition we only need to show that  $\deg[L^*(f), L^*(g)] \geq \deg[f, g]$ . But  $f = (L^{-1})^*(L^*(f))$  and  $g = (L^{-1})^*(L^*(g))$ . So applying Proposition 11 to the polynomials  $L^*(f), L^*(g)$  and the mapping  $L^{-1}$  we obtain

$$\deg[f, g] = \deg[(L^{-1})^*(L^*(f)), (L^{-1})^*(L^*(g))] \leq \deg[L^*(f), L^*(g)].$$

□

**2.3. Shestakov-Umirbaev reductions.** In this section we present the most remarkable result of Shestakov and Umirbaev, Theorem 8. The notions of reductions of types I-IV are crucial in this theorem. Thus we start with the following definitions (see [47] or [48]).

**Definition 3.** Let  $\Theta = (f_1, f_2, f_3)$  be an automorphism of  $A = \mathbb{C}[X, Y, Z]$  such that (for some  $n \in \mathbb{N}^*$ )  $\deg f_1 = 2n$ ,  $\deg f_2 = ns$ , where  $s \geq 3$  is an odd number,  $2n < \deg f_3 \leq ns$  and  $\overline{f_3} \notin \mathbb{C}[\overline{f_1}, \overline{f_2}]$ . Suppose that there exists  $\alpha \in \mathbb{C}^*$  such that the elements  $g_1 = f_1$ ,  $g_2 = f_2 - \alpha f_3$  satisfy the following conditions:

- (i)  $g_1, g_2$  is a 2-reduced pair and  $\deg g_1 = \deg f_1$ ,  $\deg g_2 = \deg f_2$ ;
- (ii) the automorphism  $(g_1, g_2, f_3)$  admits an elementary reduction  $(g_1, g_2, g_3)$  with  $\deg [g_1, g_3] < \deg g_2 + \deg [g_1, g_2]$ .

Then we will say that  $\Theta$  admits a reduction  $(g_1, g_2, g_3)$  of type I.

We will also say that a polynomial automorphism  $F = (F_1, F_2, F_3)$  admits a reduction of type I if for some permutation  $\sigma$  of  $\{1, 2, 3\}$ , the automorphism  $\Theta = (F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)})$  admits a reduction of type I.

Before proposing next definitions we present an example due to Shestakov and Umirbaev of a tame automorphism of  $\mathbb{C}^3$  which does not admit an elementary reduction but admits a reduction of type I.

**Example 1.** Let

$$\begin{aligned} T_1(x_1, x_2, x_3) &= (x_1, x_2 + x_1^2, x_3 + 2x_1x_2 + x_1^3), \\ T_2(y_1, y_2, y_3) &= (6y_1 + 6y_2y_3 + y_3^3, 4y_2 + y_3^2, y_3), \\ T_3(z_1, z_2, z_3) &= (z_1, z_2, z_3 + z_1^2 - z_2^3), \\ L(u_1, u_2, u_3) &= (u_1 + u_3, u_2, u_3) \end{aligned}$$

and

$$\begin{aligned} G &= T_3 \circ T_2 \circ T_1, \\ F &= L \circ G. \end{aligned}$$

It is easy to see that

$$\text{mdeg}(T_2 \circ T_1) = (9, 6, 3),$$

and because

$$(6y_1 + 6y_2y_3 + y_3^3)^2 - (4y_2 + y_3^2)^3 = 36y_1^2 + 72y_1y_2y_3 + 12y_1y_3^3 - 12y_2^2y_3^2 - 64y_2^3$$

and (provided that  $y_1 = x_1$ ,  $y_2 = x_2 + x_1^2$  and  $y_3 = x_3 + 2x_1x_2 + x_1^3$ )

$$\begin{aligned} &12y_1y_3^3 - 12y_2^2y_3^2 \\ &= 12x_1(x_3 + 2x_1x_2 + x_1^3)^3 - 12(x_2 + x_1^2)^2(x_3 + 2x_1x_2 + x_1^3)^2 \\ &= 12x_3x_1^7 - 12x_1^6x_2^2 + \text{lower degree monomials}, \end{aligned}$$

we have

$$\text{mdeg}(T_3 \circ T_2 \circ T_1) = (9, 6, 8)$$

and so

$$\text{mdeg} F = \text{mdeg}(L \circ G) = (9, 6, 8).$$

From the construction of  $F$  it is clear that  $F$  is a tame automorphism. Moreover, it does not admit an elementary reduction. Indeed, if we put  $F = (F_1, F_2, F_3)$

and assume that  $(F_1 - g(F_2, F_3), F_2, F_3)$ , for some  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$  then we must have

$$(10) \quad \deg g(F_2, F_3) = 9.$$

But by Proposition 9, we have

$$(11) \quad \deg g(F_2, F_3) \geq q(p \cdot 8 - 6 - 8 + \deg[F_2, F_3]) + 8r,$$

where  $\deg_Y g(X, Y) = qp + r, 0 \leq r < p, p = \frac{6}{\gcd(6, 8)} = 3$ . Thus by (10) and (11) and because  $p \cdot 8 - 6 - 8 + \deg[F_2, F_3] = 10 + \deg[F_2, F_3] \geq 12 > 9$ , we must have  $q = 0$  and  $r \leq 1$ . Thus  $g$  must be of the form

$$(12) \quad g(X, Y) = g_0(X) + g_1(X)Y.$$

Since  $8\mathbb{N} \cap (6 + 8\mathbb{N}) = \emptyset$ , from (10) and (12) we obtain  $9 = \deg g(F_2, F_3) \in 8\mathbb{N} \cup (6 + 8\mathbb{N})$ , a contradiction.

Next, if we assume that  $(F_1, F_2 - g(F_3, F_1), F_3)$ , for some  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$  then we must have

$$(13) \quad \deg g(F_3, F_1) = 6.$$

But by Proposition 9,

$$(14) \quad \deg g(F_3, F_1) \geq q(p \cdot 9 - 9 - 8 + \deg[F_3, F_1]) + 9r,$$

where  $\deg_Y g(X, Y) = qp + r, 0 \leq r < p, p = \frac{8}{\gcd(8, 9)} = 8$ . Because  $p \cdot 9 - 9 - 8 + \deg[F_3, F_1] = 55 + \deg[F_3, F_1] \geq 57 > 8$ , from (13) and (14) we obtain  $q = r = 0$ . This means that  $g(X, Y) = g(X)$  and  $\deg g(F_3, F_1) = \deg g(F_3) \in 8\mathbb{N}$ . However,  $6 \notin 8\mathbb{N}$ .

Finally, if we assume that  $(F_1, F_2, F_3 - g(F_2, F_1))$ , for some  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$  then

$$(15) \quad \deg g(F_2, F_1) = 8.$$

As before, by Proposition 9,

$$(16) \quad \deg g(F_2, F_1) \geq q(p \cdot 9 - 9 - 6 + \deg[F_2, F_1]) + 9r,$$

where  $\deg_Y g(X, Y) = qp + r, 0 \leq r < p, p = \frac{6}{\gcd(6, 9)} = 2$ . In this case  $p \cdot 9 - 9 - 6 = 3$  is not large enough for our purpose but  $\deg[F_2, F_1]$  is. Indeed,

$$\begin{aligned} \frac{\partial F_1}{\partial x_i} &= \frac{\partial u_1}{\partial x_i} + \frac{\partial u_3}{\partial x_i} \\ &= \frac{\partial z_1}{\partial x_i} + \frac{\partial z_3}{\partial x_i} + 2z_1 \frac{\partial z_1}{\partial x_i} - 3z_2^2 \frac{\partial z_2}{\partial x_i} \end{aligned}$$

and

$$\frac{\partial F_2}{\partial x_i} = \frac{\partial u_2}{\partial x_i} = \frac{\partial z_2}{\partial x_i}.$$

Thus, for  $1 \leq i < j \leq 3$ ,

$$\begin{aligned}
 \frac{\partial F_1}{\partial x_i} \frac{\partial F_2}{\partial x_j} - \frac{\partial F_1}{\partial x_j} \frac{\partial F_2}{\partial x_i} &= \left( \frac{\partial z_1}{\partial x_i} + \frac{\partial z_3}{\partial x_i} + 2z_1 \frac{\partial z_1}{\partial x_i} - 3z_2^2 \frac{\partial z_2}{\partial x_i} \right) \frac{\partial z_2}{\partial x_j} \\
 &\quad - \left( \frac{\partial z_1}{\partial x_j} + \frac{\partial z_3}{\partial x_j} + 2z_1 \frac{\partial z_1}{\partial x_j} - 3z_2^2 \frac{\partial z_2}{\partial x_j} \right) \frac{\partial z_2}{\partial x_i} \\
 (17) \qquad &= \left( \frac{\partial z_1}{\partial x_i} \frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right) + \left( \frac{\partial z_3}{\partial x_i} \frac{\partial z_2}{\partial x_j} - \frac{\partial z_3}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right) \\
 &\quad + 2z_1 \left( \frac{\partial z_1}{\partial x_i} \frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right).
 \end{aligned}$$

Since  $z_1, z_2, z_3$  are algebraically independent, by Corollary 7 for at least one pair  $i, j, 1 \leq i < j \leq 3$ , we have

$$\frac{\partial z_1}{\partial x_i} \frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j} \frac{\partial z_2}{\partial x_i} \neq 0.$$

And since  $\deg z_1 = 9$ , for that pair  $i, j$  we have

$$(18) \qquad \deg 2z_1 \left( \frac{\partial z_1}{\partial x_i} \frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right) \geq 9.$$

Of course we also have

$$(19) \qquad \deg 2z_1 \left( \frac{\partial z_1}{\partial x_i} \frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right) > \deg \left( \frac{\partial z_1}{\partial x_i} \frac{\partial z_2}{\partial x_j} - \frac{\partial z_1}{\partial x_j} \frac{\partial z_2}{\partial x_i} \right).$$

Since moreover

$$\begin{aligned}
 \frac{\partial z_2}{\partial x_i} &= 4 \frac{\partial y_2}{\partial x_i} + 2y_3 \frac{\partial y_3}{\partial x_i}, \\
 \frac{\partial z_3}{\partial x_i} &= \frac{\partial y_3}{\partial x_i}
 \end{aligned}$$

and

$$\begin{aligned}
 \deg y_2 &= \deg (x_2 + x_1^2) = 2, \\
 \deg y_3 &= \deg (x_3 + 2x_1x_2 + x_1^3) = 3,
 \end{aligned}$$

it follows that

$$\begin{aligned}
 \frac{\partial z_2}{\partial x_i} \frac{\partial z_3}{\partial x_j} - \frac{\partial z_2}{\partial x_j} \frac{\partial z_3}{\partial x_i} &= \left( 4 \frac{\partial y_2}{\partial x_i} + 2y_3 \frac{\partial y_3}{\partial x_i} \right) \frac{\partial y_3}{\partial x_j} - \left( 4 \frac{\partial y_2}{\partial x_j} + 2y_3 \frac{\partial y_3}{\partial x_j} \right) \frac{\partial y_3}{\partial x_i} \\
 &= 4 \left( \frac{\partial y_2}{\partial x_i} \frac{\partial y_3}{\partial x_j} - \frac{\partial y_2}{\partial x_j} \frac{\partial y_3}{\partial x_i} \right),
 \end{aligned}$$

and so

$$(20) \qquad \deg \left( \frac{\partial z_2}{\partial x_i} \frac{\partial z_3}{\partial x_j} - \frac{\partial z_2}{\partial x_j} \frac{\partial z_3}{\partial x_i} \right) = \deg \left( \frac{\partial y_2}{\partial x_i} \frac{\partial y_3}{\partial x_j} - \frac{\partial y_2}{\partial x_j} \frac{\partial y_3}{\partial x_i} \right) \leq 3.$$

Finally, by (17) - (20),

$$(21) \qquad \deg [F_1, F_2] \geq 11.$$

Now, using (21) and (16) we find that

$$(22) \qquad \deg g(F_2, F_1) \geq q \cdot 14 + 9r.$$

Thus, by (22) and (15), we have  $q = r = 0$ . This means that  $g(X, Y) = g(X)$  and  $\deg g(F_2, F_1) = \deg g(F_2) \in 6\mathbb{N}$ , contrary to  $8 \notin 6\mathbb{N}$ .

For more information about polynomial automorphisms which admit reductions of type I see [23].

**Definition 4.** Let  $\Theta = (f_1, f_2, f_3)$  be an automorphism of  $A = \mathbb{C}[X, Y, Z]$  such that (for some  $n \in \mathbb{N}^*$ )  $\deg f_1 = 2n$ ,  $\deg f_2 = 3n$ ,  $\frac{3}{2}n < \deg f_3 \leq 2n$  and  $\bar{f}_1, \bar{f}_3$  are linearly independent. Suppose that there exist  $\alpha, \beta \in \mathbb{C}$  with  $(\alpha, \beta) \neq (0, 0)$  such that the elements  $g_1 = f_1 - \alpha f_3$ ,  $g_2 = f_2 - \beta f_3$  satisfy the following conditions:

- (i)  $g_1, g_2$  is a 2-reduced pair and  $\deg g_1 = \deg f_1$ ,  $\deg g_2 = \deg f_2$ ;
- (ii) the automorphism  $(g_1, g_2, f_3)$  admits an elementary reduction  $(g_1, g_2, g_3)$  with  $\deg [g_1, g_3] < \deg g_2 + \deg [g_1, g_2]$ .

Then we will say that  $\Theta$  admits a reduction  $(g_1, g_2, g_3)$  of type II.

We will also say that a polynomial automorphism  $F = (F_1, F_2, F_3)$  admits a reduction of type II if for some permutation  $\sigma$  of  $\{1, 2, 3\}$ , the automorphism  $\Theta = (F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)})$  admits a reduction of type II.

**Definition 5.** Let  $\Theta = (f_1, f_2, f_3)$  be an automorphism of  $A = \mathbb{C}[X, Y, Z]$  such that (for some  $n \in \mathbb{N}^*$ )  $\deg f_1 = 2n$ , and either

$$\deg f_2 = 3n, \quad n < \deg f_3 \leq \frac{3}{2}n,$$

or

$$\frac{5}{2}n < \deg f_2 \leq 3n, \quad \deg f_3 = \frac{3}{2}n.$$

Suppose that there exist  $\alpha, \beta, \gamma \in \mathbb{C}$  such that the elements  $g_1 = f_1 - \beta f_3$ ,  $g_2 = f_2 - \gamma f_3 - \alpha f_3^2$  satisfy the following conditions:

- (i)  $g_1, g_2$  is a 2-reduced pair and  $\deg g_1 = 2n$ ,  $\deg g_2 = 3n$ ;
- (ii) there exists  $g_3$  of the form  $g_3 = \sigma f_3 + g$ , where  $\sigma \in \mathbb{C}^*$ ,  $g \in \mathbb{C}[g_1, g_2]$ , such that  $\deg g_3 \leq \frac{3}{2}n$ ,  $\deg [g_1, g_3] < 3n + \deg [g_1, g_2]$ .

If  $(\alpha, \beta, \gamma) \neq (0, 0, 0)$  and  $\deg g_3 < n + \deg [g_1, g_2]$ , then we will say that  $\Theta$  admits a reduction  $(g_1, g_2, g_3)$  of type III. On the other hand, if there exists  $\mu \in \mathbb{C}^*$  such that  $\deg (g_2 - \mu g_3^2) \leq 2n$ , then we will say that  $\Theta$  admits a reduction  $(g_1, g_2 - \mu g_3^2, g_3)$  of type IV.

We will also say that a polynomial automorphism  $F = (F_1, F_2, F_3)$  admits a reduction of type III (type IV) if for some permutation  $\sigma$  of  $\{1, 2, 3\}$ , the automorphism  $\Theta = (F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)})$  admits a reduction of type III (type IV).

Now, we can present the above mentioned theorem.

**Theorem 12.** ([47], Theorem 3) Let  $F = (F_1, F_2, F_3)$  be a tame automorphism of  $\mathbb{C}^3$ . If  $\deg F_1 + \deg F_2 + \deg F_3 > 3$  (in other words, if  $F$  is not an affine automorphism), then  $F$  admits either an elementary reduction or a reduction of one of types I-IV.

**2.4. Some number theory.** We will use the following result from number theory, connected with the so-called coin problem or Frobenius problem.

**Theorem 13.** (see e.g. [10]) If  $d_1, d_2$  are positive integers such that  $\gcd(d_1, d_2) = 1$ , then for every integer  $k \geq (d_1 - 1)(d_2 - 1)$  there are  $k_1, k_2 \in \mathbb{N}$  such that

$$k = k_1 d_1 + k_2 d_2.$$

Moreover  $(d_1 - 1)(d_2 - 1) - 1 \notin d_1 \mathbb{N} + d_2 \mathbb{N}$ .

The proof of the above theorem can be found in the number theory literature, but for the convenience of the reader we give it here. In the proof we will write  $M(d_1, d_2)$  for the minimal  $s \in \mathbb{N}$  such that  $\{s, s+1, \dots\} \subset d_1\mathbb{N} + d_2\mathbb{N}$ . Let us mention that the so-called Frobenius number (the maximal  $s \in \mathbb{N}$  such that  $s \notin d_1\mathbb{N} + d_2\mathbb{N}$ ) is equal to  $M(d_1, d_2) - 1$ .

*Proof.* Without loss of generality we can assume that  $1 < d_1 \leq d_2$ . Indeed, if  $d_1 = 1$ , then  $d_1\mathbb{N} + d_2\mathbb{N} = \mathbb{N}$  and  $(d_1 - 1)(d_2 - 1) = 0$ . Thus for any  $r = 1, \dots, d_1 - 1$  there are integers  $k_{1,r}, k_{2,r} \in \mathbb{Z}$  such that

$$k_{1,r}d_1 + k_{2,r}d_2 = r.$$

Since  $d_1, d_2, r > 0$  and  $r < d_1 \leq d_2$ , we have  $k_{1,r}k_{2,r} < 0$ . Moreover, since  $(k_{1,r} - d_2)d_1 + (k_{2,r} + d_1)d_2 = k_{1,r}d_1 + k_{2,r}d_2 = r$ , we can assume that  $k_{2,r} > 0$ . Notice that we can assume even more, namely that  $k_{2,r} > 0$  and  $k_{1,r} \geq d_2 - 1$ . Indeed, let  $k_{1,r}, k_{2,r} \in \mathbb{Z}$  be such that  $k_{1,r}d_1 + k_{2,r}d_2 = r$ ,  $k_{2,r} > 0$  and there are no  $k'_{1,r}, k'_{2,r} \in \mathbb{Z}$  such that  $k'_{1,r}d_1 + k'_{2,r}d_2 = r$ ,  $k'_{2,r} > 0$  and  $k'_{2,r} < k_{2,r}$ . Then, since  $(k_{1,r} + d_2)d_1 + (k_{2,r} - d_1)d_2 = k_{1,r}d_1 + k_{2,r}d_2 = r$ , we have  $k_{2,r} - d_1 \leq 0$  (since  $r < d_1 \leq d_2$  we actually have  $k_{2,r} - d_1 < 0$ ). Thus  $k_{1,r} + d_2 > 0$ , and so  $k_{1,r} \geq d_2 - 1$ .

It is easy to see that to show that any natural number  $k \geq (d_1 - 1)(d_2 - 1)$  is in  $d_1\mathbb{N} + d_2\mathbb{N}$ , we only need to show that

$$(d_1 - 1)(d_2 - 1), (d_1 - 1)(d_2 - 1) + 1, \dots, (d_1 - 1)(d_2 - 1) + d_1 - 1 \in d_1\mathbb{N} + d_2\mathbb{N}.$$

First,

$$\begin{aligned} (d_1 - 1)(d_2 - 1) &= (d_2 - 1)d_1 - d_2 + 1 = (d_2 - 1)d_1 - d_2 + k_{1,1}d_1 + k_{2,1}d_2 \\ &= (d_2 - 1 + k_{1,1})d_1 + (k_{2,1} - 1)d_2 \in d_1\mathbb{N} + d_2\mathbb{N}, \end{aligned}$$

because  $k_{1,1} \geq d_2 - 1$  and  $k_{2,1} > 0$ . Similarly, we show that  $(d_1 - 1)(d_2 - 1) + 1 = (d_2 - 1)d_1 - d_2 + 2, \dots, (d_1 - 1)(d_2 - 1) + d_1 - 2 = (d_2 - 1)d_1 - d_2 + (d_1 - 1) \in d_1\mathbb{N} + d_2\mathbb{N}$ . To see that  $(d_1 - 1)(d_2 - 1) + d_1 - 1 \in d_1\mathbb{N} + d_2\mathbb{N}$  we write

$$(d_1 - 1)(d_2 - 1) + d_1 - 1 = d_1d_2 - d_1 - d_2 + 1 + d_1 - 1 = (d_1 - 1)d_2.$$

Thus we have shown that  $M(d_1, d_2) \leq (d_1 - 1)(d_2 - 1)$ .

To prove that  $M(d_1, d_2) = (d_1 - 1)(d_2 - 1)$  it is enough to show that  $(d_1 - 1)(d_2 - 1) - 1 \notin d_1\mathbb{N} + d_2\mathbb{N}$ . Since  $(d_2 - 1)d_1 - d_2 = (d_1 - 1)(d_2 - 1) - 1$  and  $\text{lcm}(d_1, d_2) = d_1d_2$ , it follows that

$$\begin{aligned} \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1d_1 + k_2d_2 = (d_1 - 1)(d_2 - 1) - 1\} \\ = \{(d_2 - 1 - ld_2, ld_1 - 1) \mid l \in \mathbb{Z}\}. \end{aligned}$$

But  $\{(d_2 - 1 - ld_2, ld_1 - 1) \mid l \in \mathbb{Z}\} \cap \mathbb{N}^2 = \emptyset$ . This ends the proof.  $\square$

### 3. SOME USEFUL RESULTS

**3.1. Some simple remarks.** In this section we make some simple but useful remarks about existence of automorphisms and tame automorphisms with given multidegree.

**Proposition 14.** ([17], Prop. 2.1) *If for  $1 \leq d_1 \leq \dots \leq d_n$  there is a sequence of integers  $1 \leq i_1 < \dots < i_m \leq n$ , such that there exists an automorphism  $G$  of  $\mathbb{C}^m$  with  $\text{mdeg } G = (d_{i_1}, \dots, d_{i_m})$ , then there exists an automorphism  $F$  of  $\mathbb{C}^n$  with  $\text{mdeg } F = (d_1, \dots, d_n)$ . Moreover, if  $G$  is tame, then  $F$  can also be found tame.*



*Proof.* Without loss of generality we can assume that  $m < n$ . Let  $1 \leq j_1 < \dots < j_{n-m} \leq n$  be such that  $\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_{n-m}\} = \{1, \dots, n\}$ . Then, of course,  $\{i_1, \dots, i_m\} \cap \{j_1, \dots, j_{n-m}\} = \emptyset$ . Consider the mapping  $h = (h_1, \dots, h_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  given by

$$h_k(x_1, \dots, x_n) = \begin{cases} x_k & \text{for } k \in \{i_1, \dots, i_m\}, \\ x_k + (x_{i_1})^{d_k} & \text{for } k \in \{j_1, \dots, j_{n-m}\}. \end{cases}$$

Of course  $h$  is an automorphism of  $\mathbb{C}^n$  and  $\deg h_k = d_k$  for  $k \in \{j_1, \dots, j_{n-m}\}$ .

Consider also the mapping  $g = (g_1, \dots, g_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  given by

$$g_k(u_1, \dots, u_n) = \begin{cases} G_l(u_{i_1}, \dots, u_{i_m}) & \text{for } k = i_l, \\ u_k & \text{for } k \in \{j_1, \dots, j_{n-m}\}. \end{cases}$$

Then  $g$  is an automorphism of  $\mathbb{C}^n$  and  $\deg g_k = d_k$  for  $k \in \{i_1, \dots, i_m\}$ .

Now  $F = g \circ h$  is an automorphism of  $\mathbb{C}^n$  (tame when  $G$  is tame) with  $\text{mdeg } F = (d_1, \dots, d_n)$ .  $\square$

**Proposition 15.** ([17], Prop. 2.2) *If for a sequence of integers  $1 \leq d_1 \leq \dots \leq d_n$  there is  $i \in \{1, \dots, n\}$  such that*

$$d_i = \sum_{j=1}^{i-1} k_j d_j \quad \text{with } k_j \in \mathbb{N},$$

*then there exists a tame automorphism  $F$  of  $\mathbb{C}^n$  with  $\text{mdeg } F = (d_1, \dots, d_n)$ .*

*Proof.* Define  $h = (h_1, \dots, h_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and  $g = (g_1, \dots, g_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$  by

$$h_k(x_1, \dots, x_n) = \begin{cases} x_k & \text{for } k = i, \\ x_k + x_i^{d_k} & \text{for } k \neq i, \end{cases}$$

and

$$g_k(u_1, \dots, u_n) = \begin{cases} u_k + u_1^{k_1} \dots u_{i-1}^{k_{i-1}} & \text{for } k = i, \\ u_k & \text{for } k \neq i. \end{cases}$$

It is easy to see that  $F = g \circ h$  is a tame automorphism with  $\text{mdeg } F = (d_1, \dots, d_n)$ .  $\square$

The above proposition implies the following result.

**Corollary 16.** ([17], Cor. 2.3) *If  $1 \leq d_1 \leq \dots \leq d_n$  is a sequence of integers with  $d_1 \leq n - 1$ , then there exists a tame automorphism  $F$  of  $\mathbb{C}^n$  with  $\text{mdeg } F = (d_1, \dots, d_n)$ .*

*Proof.* Let  $r_i \in \{0, 1, \dots, d_1 - 1\}$ , for  $i = 2, \dots, n$ , be such that  $d_i \equiv r_i \pmod{d_1}$ . If there is an  $i \in \{2, \dots, n\}$  such that  $r_i = 0$ , then  $d_i = kd_1$  for some  $k \in \mathbb{N}^*$  and by Proposition 15, there exists a tame automorphism  $F$  of  $\mathbb{C}^n$  with the desired properties.

Thus assume that  $r_i \neq 0$  for all  $i = 2, \dots, n$ . Since  $d_1 - 1 < n - 1$ , there are  $i, j \in \{2, \dots, n\}$ ,  $i \neq j$ , such that  $r_i = r_j$ . Without loss of generality we can assume that  $i < j$ . Then  $d_j = d_i + kd_1$  for some  $k \in \mathbb{N}$ , and by Proposition 15 there exists a tame automorphism  $F$  of  $\mathbb{C}^n$  with the desired properties.  $\square$

The above corollary can be improved as follows.

**Theorem 17.** *If  $1 \leq d_1 \leq \dots \leq d_n$  is a sequence of integers with*

$$\frac{d_1}{\gcd(d_1, \dots, d_n)} \leq n - 1,$$

*then there exists a tame automorphism  $F$  of  $\mathbb{C}^n$  with  $\text{mdeg } F = (d_1, \dots, d_n)$ .*

*Proof.* Let  $d = \gcd(d_1, \dots, d_n)$ . Then the numbers  $r_2, \dots, r_n$  defined as in the proof of Corollary 16 satisfy  $r_i \in \{0, d, 2d, \dots, d_1 - d\}$  for  $i = 2, \dots, n$ . Since the number of elements of the set  $\{0, d, 2d, \dots, d_1 - d\}$  is equal to

$$\frac{d_1}{\gcd(d_1, \dots, d_n)} \leq n - 1,$$

we can use the same arguments as in the proof of Corollary 16.  $\square$

Combining Theorem 17 and Proposition 14 we obtain the following result.

**Corollary 18.** *If for  $1 \leq d_1 \leq \dots \leq d_n$  there is a sequence of integers  $1 \leq i_1 < \dots < i_m \leq n$ , such that*

$$\frac{d_{i_1}}{\gcd(d_{i_1}, \dots, d_{i_m})} \leq m - 1,$$

*then there exists a tame automorphism  $F$  of  $\mathbb{C}^n$  with  $\text{mdeg } F = (d_1, \dots, d_n)$ .*

**3.2. Reducibility of type I and II.** Now we will show that in our considerations we do not need to pay attention to reducibility of type I and II.

**Lemma 19.** *Let  $(d_1, d_2, d_3) \neq (1, 1, 1)$ ,  $d_1 \leq d_2 \leq d_3$  be a sequence of positive integers. If there is an automorphism (a tame automorphism)  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $F$  admits a reduction of type I or II and  $\text{mdeg } F = (d_1, d_2, d_3)$ , then there is also an automorphism (a tame automorphism)  $\tilde{F} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\tilde{F}$  admits an elementary reduction and  $\text{mdeg } \tilde{F} = (d_1, d_2, d_3)$ . Moreover, if  $F(0, 0, 0) = (0, 0, 0)$ , then  $\tilde{F}$  can also be found such that  $\tilde{F}(0, 0, 0) = (0, 0, 0)$ .*

*Proof.* Assume that  $F = (F_1, F_2, F_3)$  admits a reduction of type I. By Definition 3 there is a permutation  $\sigma$  of  $\{1, 2, 3\}$  and  $\alpha \in \mathbb{C}^*$  such that the elements  $g_1 = F_{\sigma(1)}, g_2 = F_{\sigma(2)} - \alpha F_{\sigma(3)}$  satisfy the following conditions:

- (i)  $g_1, g_2$  is a 2-reduced pair and  $\deg g_1 = \deg F_{\sigma(1)}, \deg g_2 = \deg F_{\sigma(2)}$ ;
- (ii) the automorphism  $(g_1, g_2, F_{\sigma(3)})$  admits an elementary reduction of the form  $(g_1, g_2, g_3)$ .

For simplicity of notation (and without loss of generality) we assume that  $\sigma = \text{id}_{\{1, 2, 3\}}$ . Thus we can take  $\tilde{F} = (g_1, g_2, F_3)$ .

If  $F$  admits a reduction of type II we can use a similar construction to obtain an automorphism  $\tilde{F}$ .

Since  $\tilde{F} = G \circ F$ , where

$$G : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x \\ y - \alpha z \\ z \end{Bmatrix} \in \mathbb{C}^3 \quad (\text{for type I})$$

or

$$G : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x - \alpha z \\ y - \beta z \\ z \end{Bmatrix} \in \mathbb{C}^3 \quad (\text{for type II})$$

$\tilde{F}$  is tame if and only if  $F$  is tame. It is also clear that  $\tilde{F}(0,0,0) = (0,0,0)$  when  $F(0,0,0) = (0,0,0)$ .  $\square$

The above lemma also implies the following

**Proposition 20.** *Let  $(d_1, d_2, d_3) \neq (1, 1, 1)$ ,  $d_1 \leq d_2 \leq d_3$ , be a sequence of positive integers. If there is a tame automorphism  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (d_1, d_2, d_3)$ , then there is also a tame automorphism  $\tilde{F} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\text{mdeg } \tilde{F} = (d_1, d_2, d_3)$  and  $\tilde{F}$  admits either an elementary reduction or a reduction of type III or IV. Moreover we can require that  $\tilde{F}(0,0,0) = (0,0,0)$ .*

*Proof.* Let  $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  be any tame automorphism with  $\text{mdeg } F = (d_1, d_2, d_3)$  and let  $T : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  be the translation given by

$$T : \mathbb{C}^3 \ni (x, y, z) \mapsto (x - F_1(0), y - F_2(0), z - F_3(0)) \in \mathbb{C}^3.$$

Then obviously  $T \circ F$  is a tame automorphism of  $\mathbb{C}^3$  such that  $\text{mdeg}(T \circ F) = \text{mdeg } F = (d_1, d_2, d_3)$  and  $(T \circ F)(0,0,0) = (0,0,0)$ . If  $T \circ F$  admits either an elementary reduction or a reduction of type III or IV, then we take  $\tilde{F} = T \circ F$ . And if  $T \circ F$  admits a reduction of type I or II, then we can use Lemma 19.  $\square$

In particular Proposition 20 says that reductions of type I and II are irrelevant for our considerations. To be precise we formulate the following

**Theorem 21.** *Let  $(d_1, d_2, d_3) \neq (1, 1, 1)$ ,  $d_1 \leq d_2 \leq d_3$  be a sequence of positive integers. To prove that there is no tame automorphism of  $\mathbb{C}^3$  with multidegree  $(d_1, d_2, d_3)$  it is enough to show that a (hypothetical) automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (d_1, d_2, d_3)$  admits neither an elementary reduction nor a reduction of type III or IV. Moreover, we can restrict our attention to automorphisms  $F$  with  $F(0,0,0) = (0,0,0)$ .*

To end this section, let us look again at Example 1. If  $F$  is the automorphism from that example, then  $\text{mdeg } F = (9, 6, 8)$  or  $(6, 8, 9)$  after permutation of coordinates. This automorphism does not admit an elementary reduction and admits a reduction of type I. One can easily see that (in the notation of Example 1)

$$T_2 \circ T_1 = T_3^{-1} \circ L^{-1} \circ F$$

is a reduction of type I of  $F$ . Moreover for  $\tilde{F} = L^{-1} \circ F$  we have

$$\text{mdeg } \tilde{F} = \text{mdeg } F$$

and  $T_3^{-1} \circ \tilde{F}$  is an elementary reduction of  $\tilde{F}$ .

**3.3. Reducibility of type III.** First of all notice that if  $1 \leq d_1 \leq d_2 \leq d_3$  are such that  $\text{mdeg } F = (d_1, d_2, d_3)$  for some automorphism  $F$  that admits a reduction of type III, then by Definition 5 there is  $n \in \mathbb{N}^*$  such that

$$d_{\sigma(1)} = 2n$$

and either

$$d_{\sigma(2)} = 3n, \quad n < d_{\sigma(3)} \leq \frac{3}{2}n,$$

or

$$\frac{5}{2}n < d_{\sigma(2)} \leq 3n, \quad d_{\sigma(3)} = \frac{3}{2}n$$

for some permutation  $\sigma$  of  $\{1, 2, 3\}$ . Since  $\frac{3}{2}n < 2n < \min\{\frac{5}{2}n, 3n\}$ , we must actually have

$$d_2 = 2n$$

and either

$$d_3 = 3n, \quad n < d_1 \leq \frac{3}{2}n,$$

or

$$\frac{5}{2}n < d_3 \leq 3n, \quad d_1 = \frac{3}{2}n.$$

Thus we have the following remark.

**Remark 3.** *If an automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (d_1, d_2, d_3)$ ,  $1 \leq d_1 \leq d_2 \leq d_3$ , admits a reduction of type III, then*

(1)  $2|d_2$ ,

(2)  $3|d_1$  or  $\frac{d_3}{d_2} = \frac{3}{2}$ .

Because of the remark above it is natural to consider the situation of the following lemma.

**Lemma 22.** *Let  $(d_1, d_2, d_3) \neq (1, 1, 1)$ ,  $d_1 \leq d_2 \leq d_3$  be a sequence of positive integers such that  $\frac{d_3}{d_2} = \frac{3}{2}$ . If there is an automorphism (a tame automorphism)  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $F$  admits a reduction of type III and  $\text{mdeg } F = (d_1, d_2, d_3)$ , then there is also an automorphism (a tame automorphism)  $\tilde{F} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\tilde{F}$  admits an elementary reduction and  $\text{mdeg } \tilde{F} = (d_1, d_2, d_3)$ . Moreover, if  $F(0, 0, 0) = (0, 0, 0)$ , then  $\tilde{F}$  can also be found such that  $\tilde{F}(0, 0, 0) = (0, 0, 0)$ .*

In the proof of this lemma we will use the following result.

**Lemma 23.** ([48], Corollary 4) *If an automorphism  $(g_1, g_2, g_3)$  is a reduction of type III of an automorphism  $(f_1, f_2, f_3)$ , then*

$$\deg g_1 + \deg g_2 + \deg g_3 < \deg f_1 + \deg f_2 + \deg f_3.$$

*Proof.* of Lemma 22 Assume that  $F = (F_1, F_2, F_3)$  admits a reduction of type III. By the above considerations, the conditions of Definition 5 must be satisfied for the automorphism  $\theta = (f_1, f_2, f_3) = (F_2, F_3, F_1)$ . Also by Definition 5 there are  $n \in \mathbb{N}^*$  and  $\alpha, \beta, \gamma \in \mathbb{C}, (\alpha, \beta, \gamma) \neq (0, 0, 0)$ , such that the elements  $g_1 = f_1 - \beta f_3$ ,  $g_2 = f_2 - \gamma f_3 - \alpha f_3^2$  satisfy the following conditions:

- (i)  $g_1, g_2$  is a 2-reduced pair and  $\deg g_1 = 2n, \deg g_2 = 3n$ ;
- (ii) there exists  $g_3$  of the form  $g_3 = \sigma f_3 + g$ , where  $\sigma \in \mathbb{C}^*, g \in \mathbb{C}[g_1, g_2]$ , such that  $\deg g_3 \leq \frac{3}{2}n, \deg[g_1, g_3] < 3n + \deg[g_1, g_2]$ ;
- (iii)  $\deg g_3 < n + \deg[g_1, g_2]$ .

Let us notice that apart from  $g_3 = \sigma f_3 + g$  we can also take  $\tilde{g}_3 = f_3 + \frac{1}{\sigma}g = f_3 + \tilde{g}$ , with  $\tilde{g} = \frac{1}{\sigma}g \in \mathbb{C}[g_1, g_2]$ .

Since in our situation, i.e.  $\frac{d_3}{d_2} = \frac{3}{2}$ , we have  $d_2 = 2n, d_3 = 3n$  and hence  $\deg F_2 = \deg f_1 = 2n = \deg g_1$  and  $\deg F_3 = \deg f_2 = 3n = \deg g_2$ , the lemma above yields  $\deg g_3 < \deg f_3 = \deg F_1 = d_1$ . This means that the automorphism  $(g_1, g_2, f_3)$ , and hence  $\tilde{F} = (F_1, g_1, g_2)$ , admits an elementary reduction. Of course  $\text{mdeg}(F_1, g_1, g_2) = \text{mdeg}(F_1, F_2, F_3)$ .

Since  $\tilde{F} = T_2 \circ T_1 \circ F$ , where the mappings

$$T_1 : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x \\ y - \beta x \\ z - \gamma x - \alpha x^2 \end{Bmatrix} \in \mathbb{C}^3$$

and

$$T_2 : \mathbb{C}^3 \ni \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \mapsto \begin{Bmatrix} x + \tilde{g}(y, z) \\ y \\ z \end{Bmatrix} \in \mathbb{C}^3$$

are triangular automorphisms,  $\tilde{F}$  is tame if and only if  $F$  is tame.

Since  $\deg F_1 > 0$ , also  $\deg \tilde{g} > 0$ , and hence  $\tilde{g} = \overline{\tilde{g} - a}$  for all  $a \in \mathbb{C}$ . Thus we can assume that  $\tilde{g}(0, 0) = 0$ . Then  $\tilde{F}(0, 0, 0) = (0, 0, 0)$  when  $F(0, 0, 0) = (0, 0, 0)$ .  $\square$

By Lemma 22 we also have the following result.

**Proposition 24.** *Let  $(d_1, d_2, d_3) \neq (1, 1, 1)$ ,  $d_1 \leq d_2 \leq d_3$ , be a sequence of positive integers such that  $\frac{d_3}{d_2} = \frac{3}{2}$ . If there is a tame automorphism  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\text{mdeg } F = (d_1, d_2, d_3)$ , then there is also a tame automorphism  $\tilde{F} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\tilde{F}$  admits either a reduction of type IV or an elementary reduction and  $\text{mdeg } \tilde{F} = (d_1, d_2, d_3)$ . Moreover we can require that  $\tilde{F}(0, 0, 0) = (0, 0, 0)$ .*

*Proof.* As in the proof of Proposition 20, we consider the automorphism  $T \circ F$ . Then we have three cases: (I)  $T \circ F$  admits a reduction of type IV or an elementary reduction; (II)  $T \circ F$  admits reduction of type III; (III)  $T \circ F$  admits a reduction of type I or II. In the first case we put  $\tilde{F} = T \circ F$ , in the second case we use Lemma 22 and in the third case we use Lemma 19.  $\square$

The above proposition means that whenever  $\frac{d_3}{d_2} = \frac{3}{2}$ , reductions of type I, II and III are irrelevant for our considerations. More precisely, we have the following

**Theorem 25.** *Let  $(d_1, d_2, d_3) \neq (1, 1, 1)$ ,  $d_1 \leq d_2 \leq d_3$ , be a sequence of positive integers such that  $\frac{d_3}{d_2} = \frac{3}{2}$  or  $3 \nmid d_1$ . To prove that there is no tame automorphism of  $\mathbb{C}^3$  with multidegree  $(d_1, d_2, d_3)$  it is enough to show that a (hypothetical) automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (d_1, d_2, d_3)$  admits neither a reduction of type IV nor an elementary reduction. Moreover, we can restrict our attention to automorphisms  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $F(0, 0, 0) = (0, 0, 0)$ .*

*Proof.* Take any  $\tilde{F} \in \text{Tame}(\mathbb{C}^3)$  with  $\text{mdeg } \tilde{F} = (d_1, d_2, d_3)$ . By Theorem 21 we can assume that  $\tilde{F}$  admits either an elementary reduction or a reduction of type III or IV.

If  $\tilde{F}$  admits a reduction of type III, then by Remark 3 and by the assumptions we have  $\frac{d_3}{d_2} = \frac{3}{2}$ . Thus we can use Proposition 24.  $\square$

**3.4. Reducibility of type IV and Kuroda's result.** In the previous sections we have proved that from our point of view reductions of type I and II are irrelevant. The same is true for reductions of type III under an additional assumption (see Theorem 25).

The following result due to Kuroda says that reduction of type IV is also irrelevant for our aim.

**Theorem 26.** ([24], Thm. 7.1) *No tame automorphism of  $\mathbb{C}^3$  admits a reduction of type IV.*

Thus we have the following

**Theorem 27.** *Let  $(d_1, d_2, d_3) \neq (1, 1, 1)$ ,  $d_1 \leq d_2 \leq d_3$ , be a sequence of positive integers. To prove that there is no tame automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F =$*

$(d_1, d_2, d_3)$  it is enough to show that a (hypothetical) automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (d_1, d_2, d_3)$  admits neither a reduction of type III nor an elementary reduction. Moreover, if we additionally assume that  $\frac{d_3}{d_2} = \frac{3}{2}$  or  $3 \nmid d_1$ , then it is enough to show that no (hypothetical) automorphism of  $\mathbb{C}^3$  with multidegree  $(d_1, d_2, d_3)$  admits an elementary reduction. In both cases we can restrict our attention to automorphisms  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $F(0, 0, 0) = (0, 0, 0)$ .

*Proof.* The proof is similar to the proof of Theorem 25.  $\square$

**3.5. Reducibility and linear change of coordinates.** Now we make some remarks that will be useful in considerations of some special cases. The main result of this section says that we can restrict our attention to the automorphisms whose linear part is the identity map.

**Lemma 28.** *If an automorphism  $(F_1, F_2, F_3)$  admits an elementary reduction, then so does  $(F_1, F_2, F_3) \circ L$  for every  $L \in GL_3(\mathbb{C})$ .*

*Proof.* Without loss of generality we can assume that  $(F_1, F_2, F_3)$  admits an elementary reduction of the form  $(F_1 - G(F_2, F_3), F_2, F_3)$ . It is easy to see that  $(F_1 \circ L - G(F_2 \circ L, F_3 \circ L), F_2 \circ L, F_3 \circ L) = (F_1 - G(F_2, F_3), F_2, F_3) \circ L$  is an elementary reduction of  $(F_1, F_2, F_3) \circ L = (F_1 \circ L, F_2 \circ L, F_3 \circ L)$ .  $\square$

We also have the following obvious lemma.

**Lemma 29.** *For every mapping  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and every  $L \in GL_n(\mathbb{C})$  we have*

$$\text{mdeg}(F \circ L) = \text{mdeg } F.$$

Combining the above two lemmas we obtain the following result.

**Theorem 30.** *For every sequence of positive integers  $(d_1, \dots, d_n) \neq (1, \dots, 1)$ , if there is a tame automorphism  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that  $F$  admits an elementary reduction,  $F(0, \dots, 0) = (0, \dots, 0)$  and  $\text{mdeg } F = (d_1, \dots, d_n)$ , then there is also a tame automorphism  $\tilde{F} : \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that  $\tilde{F}$  admits an elementary reduction,  $\text{mdeg } \tilde{F} = (d_1, \dots, d_n)$ ,  $\tilde{F}(0, \dots, 0) = (0, \dots, 0)$  and the linear part of  $\tilde{F}$  is equal to  $\text{id}_{\mathbb{C}^n}$ .*

*Proof.* Let  $L$  be the linear part of  $F$ . Since  $F \in \text{Aut}(\mathbb{C}^n)$ , we have  $L \in GL_n(\mathbb{C})$ . The linear part of  $F \circ L^{-1}$  is equal to  $\text{id}_{\mathbb{C}^n}$ . We also have  $(F \circ L^{-1})(0, \dots, 0) = F(0, \dots, 0) = (0, \dots, 0)$ .  $\square$

**3.6. Relationship between the degree of the Poisson bracket and the number of variables.** The main result of this section is Lemma 32 below. We start with the following

**Lemma 31.** *Let  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  be such that*

$$f = X_1 + f_2 + \dots + f_l, \quad g = X_2 + g_2 + \dots + g_m,$$

*where  $f_i, g_i$  are homogeneous forms of degree  $i$ . If  $\deg[f, g] = 2$  and  $f$  does not involve  $X_i$ , where  $i > 2$ , then  $g$  does not involve  $X_i$  either.*

*Proof.* The assumption  $\deg[f, g] = 2$  implies that for all  $1 \leq k < l \leq n$  we have

$$\deg \text{Jac}^{X_k X_l}(f, g) \leq 0.$$

In particular,

$$\deg \text{Jac}^{X_1 X_i}(f, g) \leq 0,$$

but

$$Jac^{X_1 X_i}(f, g) = \frac{\partial f}{\partial X_1} \frac{\partial g}{\partial X_i} - \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_1} = \frac{\partial f}{\partial X_1} \frac{\partial g}{\partial X_i}.$$

Thus  $\deg \frac{\partial g}{\partial X_i} \leq 0$ . In other words if  $g$  involves  $X_i$  then  $X_i$  occurs in the linear part of  $g$ . But this contradicts the assumptions.  $\square$

Now we are in a position to prove the following lemma that is one of the main ingredients in proving, for instance, that  $(5, 6, 9) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .

**Lemma 32.** *Let  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  be such that*

$$f = X_1 + f_2 + \dots + f_l, \quad g = X_2 + g_2 + \dots + g_m,$$

*where  $f_i, g_i$  are homogeneous forms of degree  $i$ . If  $\deg[f, g] = 2$ , then  $f, g \in \mathbb{C}[X_1, X_2]$ .*

*Proof.* Without loss of generality we can assume that  $l \leq m$ . Let  $i > 2$  be arbitrary. Let us notice that

$$[Jac^{X_1 X_i}(f, g)]_1 = Jac^{X_1 X_i}(X_1, g_2) + Jac^{X_1 X_i}(f_2, X_2) = \frac{\partial g_2}{\partial X_i}$$

and

$$[Jac^{X_2 X_i}(f, g)]_1 = Jac^{X_2 X_i}(X_1, g_2) + Jac^{X_2 X_i}(f_2, X_2) = -\frac{\partial f_2}{\partial X_i},$$

where  $[Jac^{X_k X_i}(f, g)]_d$  is the homogeneous part of degree  $d$  of  $Jac^{X_k X_i}(f, g)$ . But the assumption  $\deg[f, g] = 2$  means in particular that  $[Jac^{X_1 X_i}(f, g)]_1 = 0$  and  $[Jac^{X_2 X_i}(f, g)]_1 = 0$ . Thus we obtain

$$\frac{\partial g_2}{\partial X_i} = 0, \quad \frac{\partial f_2}{\partial X_i} = 0,$$

and so  $f_2, g_2$  do not involve  $X_i$ . It follows that

$$\begin{aligned} [Jac^{X_1 X_i}(f, g)]_2 &= Jac^{X_1 X_i}(X_1, g_3) + Jac^{X_1 X_i}(f_2, g_2) + Jac^{X_1 X_i}(f_3, X_2) \\ &= Jac^{X_1 X_i}(X_1, g_3) = \frac{\partial g_3}{\partial X_i} \end{aligned}$$

and

$$\begin{aligned} [Jac^{X_2 X_i}(f, g)]_2 &= Jac^{X_2 X_i}(X_1, g_3) + Jac^{X_2 X_i}(f_2, g_2) + Jac^{X_2 X_i}(f_3, X_2) \\ &= Jac^{X_2 X_i}(f_3, X_2) = -\frac{\partial f_3}{\partial X_i}. \end{aligned}$$

Since  $\deg[f, g] = 2$  implies  $[Jac^{X_1 X_i}(f, g)]_2 = 0$  and  $[Jac^{X_2 X_i}(f, g)]_2 = 0$ , we see that

$$\frac{\partial g_3}{\partial X_i} = 0, \quad \frac{\partial f_3}{\partial X_i} = 0,$$

and so  $f_3, g_3$  do not involve  $X_i$ .

Proceeding inductively, when we know that  $f_2, \dots, f_{l-1}, g_2, \dots, g_{l-1}$  do not involve  $X_i$ , we obtain

$$\begin{aligned} [Jac^{X_1 X_i}(f, g)]_{n-1} &= Jac^{X_1 X_i}(X_1, g_n) + \dots + Jac^{X_1 X_i}(f_n, X_2) \\ &= Jac^{X_1 X_i}(X_1, g_n) = \frac{\partial g_n}{\partial X_i} \end{aligned}$$

and

$$\begin{aligned} [Jac^{X_2 X_i}(f, g)]_{n-1} &= Jac^{X_2 X_i}(X_1, g_n) + \cdots + Jac^{X_2 X_i}(f_n, X_2) \\ &= Jac^{X_2 X_i}(f_n, X_2) = -\frac{\partial f_n}{\partial X_i}. \end{aligned}$$

By the assumption  $\deg[f, g] = 2$ , as before we find that  $f_n$  and  $g_n$  do not involve  $X_i$ . Therefore  $f$  does not involve  $X_i$ . To deduce that  $g$  does not involve  $X_i$  either, we can use Lemma 31.  $\square$

By similar arguments one can prove the following

**Theorem 33.** *Let  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  be such that*

$$f = X_1 + f_2 + \cdots + f_l, \quad g = X_2 + g_2 + \cdots + g_m,$$

*where  $f_i, g_i$  are homogeneous forms of degree  $i$ . If  $\deg[f, g] = d \leq \min\{k, m\}$ ,  $d \geq 2$ , and  $f_i, g_i$  for  $i = 1, \dots, d-1$  do not involve  $X_l$ , where  $l > 2$ , then  $f$  and  $g$  do not involve  $X_l$ .*

The results of Lemma 32 and Theorem 33 can be generalized as follows.

**Theorem 34.** *Let  $f, g \in \mathbb{C}[X_1, \dots, X_n]$  be such that*

$$f = f_1 + f_2 + \cdots + f_l, \quad g = g_1 + g_2 + \cdots + g_m,$$

*where  $f_i, g_i$  are homogeneous forms of degree  $i$ . If  $f_1, g_1$  are linearly independent,  $\deg[f, g] = d \leq \min\{l, m\}$ ,  $d \geq 2$ , and  $f_i, g_i$ , for  $i = 1, \dots, d-1$ , do not involve  $X_r$ , where  $r > 2$ , then  $f$  and  $g$  do not involve  $X_r$ .*

*Proof.* Let  $l_3, \dots, l_{n-1} \in \mathbb{C}[X_1, \dots, X_{r-1}, X_{r+1}, \dots, X_n]$  be linear forms such that  $f_1, g_1, l_3, \dots, l_{n-1}$  are linearly independent. Then  $f_1, g_1, l_3, \dots, l_{n-1}, X_r$  are also linearly independent. Let  $L = (f_1, g_1, l_3, \dots, l_{n-1}, X_r) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ . Of course  $L, L^{-1} \in GL_n(\mathbb{C})$ , and by Lemma 10,  $\deg[f \circ L^{-1}, g \circ L^{-1}] = \deg[f, g] = d$ . One can also check that  $(f \circ L^{-1})_1 = X_1$ ,  $(g \circ L^{-1})_1 = X_2$  and that  $(f \circ L^{-1})_i, (g \circ L^{-1})_i$ , for  $i = 1, \dots, d-1$ , do not involve  $X_l$ . Thus by Theorem 33,  $f \circ L^{-1}, g \circ L^{-1}$  do not involve  $X_l$  either. And one can easily check that the same is true for  $f = (f \circ L^{-1}) \circ L$  and  $g = (g \circ L^{-1}) \circ L$ .  $\square$



4. THE CASE  $(p_1, p_2, d_3)$  AND ITS GENERALIZATION

4.1. **The case  $(p_1, p_2, d_3)$ .** Here we investigate the set

$$\{(p_1, p_2, d_3) : 3 \leq p_1 < p_2 \leq d_3, p_1, p_2 \text{ prime numbers}\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)).$$

The complete description of this set is given in the following theorem.

**Theorem 35.** ([18], *Thm. 1.1*) *Let  $d_3 \geq p_2 > p_1 \geq 3$  be positive integers. If  $p_1$  and  $p_2$  are prime numbers, then  $(p_1, p_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $d_3 \in p_1\mathbb{N} + p_2\mathbb{N}$ .*

*Proof.* If  $d_3 \in p_1\mathbb{N} + p_2\mathbb{N}$ , then by Proposition 15, there exists a tame automorphism  $F \in \text{Tame}(\mathbb{C}^3)$  such that  $\text{mdeg } F = (p_1, p_2, d_3)$ . Thus in order to prove the theorem we must only show that if  $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$ , then there is no tame automorphism  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (p_1, p_2, d_3)$ . Thus up to the end of the proof we assume that  $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$ .

Assume, to the contrary, that there are tame automorphisms  $F$  of  $\mathbb{C}^3$  such that  $\text{mdeg } F = (p_1, p_2, d_3)$ . By Theorem 27, we only need to show that all such automorphisms do not admit an elementary reduction and reduction of type III. Since  $p_2 > 3$  is a prime number,  $2 \nmid p_2$ . Hence by Remark 3, no automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (p_1, p_2, d_3)$  admits a reduction of type III.

Assume, to the contrary, that there is an automorphism  $F = (F_1, F_2, F_3)$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (p_1, p_2, d_3)$  that admits an elementary reduction. Notice that, by Theorem 13,

$$(23) \quad d_3 < (p_1 - 1)(p_2 - 1).$$

Assume that

$$(F_1, F_2, F_3 - g(F_1, F_2)),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Then we have  $\deg g(F_1, F_2) = \deg F_3 = d_3$ . But, by Proposition 9,

$$\deg g(F_1, F_2) \geq q(p_1 p_2 - p_1 - p_2 + \deg[F_1, F_2]) + r p_2,$$

where  $\deg_Y g(X, Y) = q p_1 + r$  with  $0 \leq r < p_1$ . Since  $F_1, F_2$  are algebraically independent,  $\deg[F_1, F_2] \geq 2$  and so

$$p_1 p_2 - p_1 - p_2 + \deg[F_1, F_2] \geq p_1 p_2 - p_1 - p_2 + 2 > (p_1 - 1)(p_2 - 1).$$

This and (23) imply that  $q = 0$ , and that:

$$g(X, Y) = \sum_{i=0}^{p_1-1} g_i(X) Y^i.$$

Since  $\text{lcm}(p_1, p_2) = p_1 p_2$ , the sets

$$p_1\mathbb{N}, p_2 + p_1\mathbb{N}, \dots, (p_1 - 1)p_2 + p_1\mathbb{N}$$

are pairwise disjoint. This yields:

$$\deg \left( \sum_{i=0}^{p_1-1} g_i(F_1) F_2^i \right) = \max_{i=0, \dots, p_1-1} (\deg F_1 \deg g_i + i \deg F_2),$$

and so

$$d_3 = \deg g(F_1, F_2) \in \bigcup_{r=0}^{p_1-1} (r p_2 + p_1\mathbb{N}) \subset p_1\mathbb{N} + p_2\mathbb{N},$$

a contradiction.

Now, assume that

$$(F_1, F_2 - g(F_1, F_3), F_3),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $F = (F_1, F_2, F_3)$ . Since  $d_3 \notin p_1\mathbb{N} + p_2\mathbb{N}$ ,  $p_1 \nmid d_3$  and  $\gcd(p_1, d_3) = 1$ . This means, by Proposition 9, that

$$\deg g(F_1, F_3) \geq q(p_1 d_3 - d_3 - p_1 + \deg[F_1, F_3]) + r d_3,$$

where  $\deg_Y g(X, Y) = qp_1 + r$  with  $0 \leq r < p_1$ . Since  $p_1 d_3 - d_3 - p_1 + \deg[F_1, F_3] \geq p_1 d_3 - 2d_3 \geq d_3 > p_2$  and since we want to have  $\deg g(F_1, F_3) = p_2$ , we conclude that  $q = r = 0$ . This means that  $g(X, Y) = g(X)$ , and then  $p_2 = \deg g(F_1) \in p_1\mathbb{N}$ , a contradiction.

Finally, if we assume that  $(F_1 - g(F_2, F_3), F_2, F_3)$ , where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ , then in the same way as in the previous case we obtain a contradiction.  $\square$

**Corollary 36.** *The following equality holds true*

$$\begin{aligned} & \{(p_1, p_2, d_3) : 3 \leq p_1 < p_2 \leq d_3, \ p_1, p_2 \text{ prime numbers}\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)) \\ &= \{(p_1, p_2, d_3) : 3 \leq p_1 < p_2 \leq d_3, \ p_1, p_2 \text{ prime numbers}, \ d_3 \in p_1\mathbb{N} + p_2\mathbb{N}\} \end{aligned}$$

#### 4.2. Some consequences.

**Theorem 37.** ([18], Thm. 3.1) *Let  $p_2 > 3$  be a prime number and  $d_3 \geq p_2$  be an integer. Then  $(3, p_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $d_3 \notin \{2p_2 - 3k \mid k = 1, \dots, \lfloor \frac{p_2}{3} \rfloor\}$ .*

*Proof.* Since  $p_2 > 3$  is a prime number,  $p_2 \equiv r \pmod{3}$  for some  $r \in \{1, 2\}$ . It is easy to see that if  $d_3 \geq p_2$  and  $d_3 \equiv 0 \pmod{3}$  or  $d_3 \equiv r \pmod{3}$ , then  $d_3 \in 3\mathbb{N} + p_2\mathbb{N}$ . Thus, by Theorem 13,

$$2(p_2 - 1) - 1 \neq 0, r \pmod{3}.$$

Take any  $d_3$  such that  $p_2 \leq d_3 \leq 2p_2 - 3$  and  $d_3 \not\equiv 0, r \pmod{3}$ . Since  $d_3 \leq 2p_2 - 3$  and  $d_3 \equiv 2p_2 - 3 \pmod{3}$ , we see that  $d_3 \notin 3\mathbb{N} + p_2\mathbb{N}$ , because otherwise we would have  $2p_2 - 3 \in 3\mathbb{N} + p_2\mathbb{N}$ , contrary to Theorem 13. Thus

$$\begin{aligned} \{d_3 \in \mathbb{N} \mid d_3 \geq p_2, d_3 \notin 3\mathbb{N} + p_2\mathbb{N}\} &= \\ &= \{d_3 \in \mathbb{N} \mid p_2 \leq d_3 \leq 2p_2 - 3, d_3 \equiv 2p_2 - 3 \pmod{3}\} \\ &= \{2p_2 - 3k \mid k = 1, \dots, \lfloor \frac{p_2}{3} \rfloor\} \end{aligned}$$

$\square$

**Theorem 38.** ([18], Thm. 3.2) (a) *If  $d_3 \geq 7$ , then  $(5, 7, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if*

$$d_3 \neq 8, 9, 11, 13, 16, 18, 23.$$

(b) *If  $d_3 \geq 11$ , then  $(5, 11, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if*

$$d_3 \neq 12, 13, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39.$$

(c) *If  $d_3 \geq 13$ , then  $(5, 13, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if*

$$d_3 \neq 14, 16, 17, 19, 21, 22, 24, 27, 29, 32, 34, 37, 42, 47.$$

(d) If  $d_3 \geq 11$ , then  $(7, 11, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if

$$d_3 \neq 12, 13, 15, 16, 17, 19, 20, 23, 24, 26, 27, 30, 31, 34, 37, 38, 41, 45, 48, 52, 59.$$

*Proof.* This is a consequence of Theorems 13 and 35. For example to prove (a), by Theorems 13 and 35 we only have to check which numbers among  $7, 8, \dots, 23 = (5-1)(7-1)-1$  are elements of the set  $5\mathbb{N} + 7\mathbb{N}$ .  $\square$

**4.3. Generalization.** Here we make a generalization of Theorem 35.

**Theorem 39.** ([20], Thm. 2.1) *Let  $d_3 \geq d_2 > d_1 \geq 3$  be positive integers. If  $d_1$  and  $d_2$  are odd numbers such that  $\gcd(d_1, d_2) = 1$ , then  $(d_1, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$ .*

*Proof.* The proof is a modification of the proof of Theorem 35. As before, if we assume that  $d_3 \in d_1\mathbb{N} + d_2\mathbb{N}$ , then by Proposition 15, there is a tame automorphism  $F$  of  $\mathbb{C}^3$  such that  $\text{mdeg } F = (d_1, d_2, d_3)$ .

Moreover, as in the proof of Theorem 35, we only need to show that no automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (d_1, d_2, d_3)$  admits an elementary reduction, when  $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$ . Assume, to the contrary, that  $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$  and that  $F = (F_1, F_2, F_3)$  is an automorphism of  $\mathbb{C}^3$  with  $\text{mdeg } F = (d_1, d_2, d_3)$  that admits an elementary reduction.

If we assume that  $(F_1, F_2, F_3 - g(F_1, F_2))$ , where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ , then we can proceed exactly in the same way as in the proof of Theorem 35.

Only in the case of an elementary reduction of the form  $(F_1, F_2 - g(F_1, F_3), F_3)$  and  $(F_1 - g(F_2, F_3), F_2, F_3)$  we must modify the arguments.

Assume that

$$(F_1, F_2 - g(F_1, F_3), F_3),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Since  $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$ , we have  $d_1 \nmid d_3$ . This follows that

$$p = \frac{d_1}{\gcd(d_1, d_3)} > 1.$$

Since  $d_1$  is odd, we also have  $p \neq 2$ . Thus by Proposition 9,

$$\deg g(F_1, F_3) \geq q(pd_3 - d_3 - d_1 + \deg[F_1, F_3]) + rd_3,$$

where  $\deg_Y g(X, Y) = qp + r$  with  $0 \leq r < p$ . Since  $p \geq 3$ , we see that  $pd_3 - d_3 - d_1 + \deg[F_1, F_3] \geq 2d_3 - d_1 + 2 > d_3$ . Since we want to have  $\deg g(F_1, F_3) = d_2$ , it follows that  $q = r = 0$ , and then  $g(X, Y) = g(X)$ . This means that  $d_2 = \deg g(F_1) \in d_1\mathbb{N}$ , contradicting to  $\gcd(d_1, d_2) = 1$  and  $1 < d_1$ .

Finally, if we assume that  $(F_1 - g(F_2, F_3), F_2, F_3)$ , where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ , then in the same way as in the previous case we obtain a contradiction.  $\square$

**4.4. The set  $\text{mdeg}(\text{Aut}(\mathbb{C}^3)) \setminus \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .** In this paragraph we say a few words about a relation between  $\text{mdeg}(\text{Tame}(\mathbb{C}^3))$  and  $\text{mdeg}(\text{Aut}(\mathbb{C}^3))$ . The obvious relation is

$$\text{mdeg}(\text{Tame}(\mathbb{C}^3)) \subset \text{mdeg}(\text{Aut}(\mathbb{C}^3))$$

and, more generally,

$$\text{mdeg}(\text{Tame}(\mathbb{C}^n)) \subset \text{mdeg}(\text{Aut}(\mathbb{C}^n)).$$

The question is, whether the set  $\text{mdeg}(\text{Tame}(\mathbb{C}^n))$  is a proper subset of  $\text{mdeg}(\text{Aut}(\mathbb{C}^n))$ . In dimension two the answer is negative due to Jung [9] and van der Kulk [21]. Namely we have

$$\text{mdeg}(\text{Tame}(\mathbb{C}^2)) = \text{mdeg}(\text{Aut}(\mathbb{C}^2)) = \{(d_1, d_2) : d_1|d_2 \text{ or } d_2|d_1\}.$$

Let us notice that the result of Shestakov and Umirbaev [48] about wildness of the Nagata's example does not imply the positive answer in dimension three. The problem is that the Nagata's example is of multidegree  $(5, 3, 1) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ . In spite of that, the answer is positive. We will show it in this subsection. Actually we show not only that the difference  $\text{mdeg}(\text{Aut}(\mathbb{C}^3)) \setminus \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  is not empty, but also that this set has infinitely many elements.

Let

$$N : \mathbb{C}^3 \ni (x, y, z) \mapsto (x + 2y(y^2 + zx) - z(y^2 + zx)^2, y - z(y^2 + zx), z) \in \mathbb{C}^3$$

be the Nagata's example and let

$$T : \mathbb{C}^3 \ni (x, y, z) \mapsto (z, y, x) \in \mathbb{C}^3.$$

We start with the following lemma.

**Lemma 40.** ([20], Lem. 3.1) *For all  $n \in \mathbb{N}$  we have  $\text{mdeg}(T \circ N)^n = (4n - 3, 4n - 1, 4n + 1)$ .*

*Proof.* We have  $T \circ N(x, y, z) = (z, y - z(y^2 + zx), x + 2y(y^2 + zx) - z(y^2 + zx)^2)$ , so the above equality is true for  $n = 1$ . Let  $(f_n, g_n, h_n) = (T \circ N)^n$  for  $f_n, g_n, h_n \in \mathbb{C}[X, Y, Z]$ . One can see that  $g_1^2 + h_1 f_1 = Y^2 + ZX$ , and by induction that  $g_n^2 + h_n f_n = Y^2 + ZX$  for any  $n \in \mathbb{N}^*$ . Thus

$$\begin{aligned} (f_{n+1}, g_{n+1}, h_{n+1}) &= (T \circ N) \circ (f_n, g_n, h_n) \\ &= (h_n, g_n - h_n(g_n^2 + h_n f_n), f_n + 2h_n(g_n^2 + h_n f_n) - h_n(g_n^2 + h_n f_n)^2) \\ &= (h_n, g_n - h_n(Y^2 + ZX), f_n + 2h_n(Y^2 + ZX) - h_n(Y^2 + ZX)^2). \end{aligned}$$

So if we assume that  $\text{mdeg}(f_n, g_n, h_n) = (4n - 3, 4n - 1, 4n + 1)$ , we obtain  $\text{mdeg}(f_{n+1}, g_{n+1}, h_{n+1}) = (4n + 1, (4n + 1) + 2, (4n + 1) + 2 \cdot 2) = (4(n + 1) - 3, 4(n + 1) - 1, 4(n + 1) + 1)$ .  $\square$

By the above lemma and Theorem 39 we obtain the following

**Theorem 41.** ([20], Thm. 3.2) *For every  $n \in \mathbb{N}$  the automorphism  $(T \circ N)^n$  is wild.*

*Proof.* For  $n = 1$  this is the result of Shestakov and Umirbaev [47, 48]. So we can assume that  $n \geq 2$ . The numbers  $4n - 3, 4n - 1$  are odd and  $\gcd(4n - 3, 4n - 1) = \gcd(4n - 3, 2) = 1$ . Since  $4n - 3 > 2$ , we see that  $4n + 1 \notin (4n - 3)\mathbb{N} + (4n - 1)\mathbb{N}$ . Then, by Theorem 39,  $(4n - 3, 4n - 1, 4n + 1) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  for  $n > 1$ . This proves that  $(T \circ N)^n$  is not a tame automorphism.  $\square$

Let us notice that in the proof of the above theorem we have also proved that

$$\{(4n - 3, 4n - 1, 4n + 1) : n \in \mathbb{N}, n \geq 2\} \subset \text{mdeg}(\text{Aut}(\mathbb{C}^3)) \setminus \text{mdeg}(\text{Tame}(\mathbb{C}^3)).$$

This gives the following result.

**Theorem 42.** ([20], *Thm. 1.1*) *The set  $\text{mdeg}(\text{Aut}(\mathbb{C}^3)) \setminus \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  is infinite.*

### 5. THE CASE $(3, d_2, d_3)$

In this section we give a complete description of the set

$$\{(3, d_2, d_3) \mid 3 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)).$$

This description is given by the following

**Theorem 43.** ([19], *Thm. 1.1*) *If  $3 \leq d_2 \leq d_3$ , then  $(3, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $3 \mid d_2$  or  $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$ .*

*Proof.* By Corollary 15, if  $3 \mid d_2$  or  $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$ , there exists a tame automorphism  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\text{mdeg } F = (3, d_2, d_3)$ . Thus in order to prove Theorem 43 it is enough to show that if  $3 \nmid d_2$  and  $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$ , then there is no tame automorphism of  $\mathbb{C}^3$  with multidegree  $(3, d_2, d_3)$ . So from now we will assume that  $3 \nmid d_2$  and  $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$ .

Since  $3 \nmid d_2$ , we have  $\gcd(3, d_2) = 1$ . Hence Theorem 13 implies that for all  $k \geq (3-1)(d_2-1) = 2d_2-2$  we have  $k \in 3\mathbb{N} + d_2\mathbb{N}$ . Thus, since  $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$ , we have

$$(24) \quad d_3 < 2d_2 - 2.$$

By Theorem 27 it is enough to show that all automorphisms  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (3, d_2, d_3)$  do not admit an elementary reduction and reduction of type III. Notice also that, since  $d_1 = 3$  and  $d_2$  can be an even number, we can not use Remark 3 to obtain that all automorphisms  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (3, d_2, d_3)$  do not admit reduction of type III.

Assume that an automorphism  $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (3, d_2, d_3)$  admits a reduction of type III. Then by Definition 5 there is a permutation  $\sigma$  of  $\{1, 2, 3\}$  and  $n \in \mathbb{N}^*$  such that  $\deg F_{\sigma(1)} = 2n$ , and either:

$$(25) \quad \deg F_{\sigma(2)} = 3n, \quad n < \deg F_{\sigma(3)} \leq \frac{3}{2}n$$

or

$$(26) \quad \frac{5}{2}n < \deg F_{\sigma(2)} \leq 3n, \quad \deg F_{\sigma(3)} = \frac{3}{2}n.$$

Since  $\frac{3}{2}n < 2n < \min\{\frac{5}{2}n, 3n\}$ , we have  $d_2 = 2n$  and either:

$$d_3 = 3n, \quad n < 3 \leq \frac{3}{2}n$$

or

$$\frac{5}{2}n < d_3 \leq 3n, \quad 3 = \frac{3}{2}n.$$

Thus  $n = 2$  and then  $5 < d_3 \leq 6$ . From the last inequalities we obtain  $d_3 = 6$ . This contradicts  $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$ .

Now, assume that  $(F_1, F_2, F_3 - g(F_1, F_2))$ , where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Hence we have  $\deg g(F_1, F_2) = \deg F_3 = d_3$ . Since  $\gcd(3, d_2) = 1$ , by Proposition 9 we have

$$\deg g(F_1, F_2) \geq q(3d_2 - d_2 - 3 + \deg[F_1, F_2]) + rd_2,$$

where  $\deg_Y g(X, Y) = 3q + r$  with  $0 \leq r < 3$ . Since  $F_1, F_2$  are algebraically independent,  $\deg[F_1, F_2] \geq 2$  and so  $3d_2 - d_2 - 3 + \deg[F_1, F_2] \geq 2d_2 - 1$ . Then (24) implies

$q = 0$ . Also by (24) we must have  $r < 2$ . Thus  $g(X, Y) = g_0(X) + g_1(X)Y$ . Since  $3\mathbb{N} \cap (d_2 + 3\mathbb{N}) = \emptyset$ , we deduce that  $\deg g(F_1, F_2) \in 3\mathbb{N} \cup (d_2 + 3\mathbb{N}) \subset 3\mathbb{N} + d_2\mathbb{N}$ , contrary to assumption.

Now, assume that  $(F_1, F_2 - g(F_1, F_3), F_3)$ , where  $g \in \mathbb{C}[X, Y]$ , there is an elementary reduction of  $(F_1, F_2, F_3)$ . Then  $\deg g(F_1, F_3) = d_2$ . Since  $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$ , it follows that  $\gcd(3, d_3) = 1$ . Then by Proposition 9 we have

$$\deg g(F_1, F_3) \geq q(3d_3 - d_3 - 3 + \deg[F_1, F_3]) + rd_3,$$

where  $\deg_Y g(X, Y) = 3q + r$  with  $0 \leq r < 3$ . Since  $3d_3 - d_3 - 3 + \deg[F_1, F_3] \geq 2d_3 - 1 > d_2$ , we infer that  $q = 0$ . Since also  $d_3 > d_2$  (because  $d_3 \geq d_2$  and  $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$ ), we see that  $r = 0$ . Thus  $g(X, Y) = g(X)$ , and  $\deg g(F_1, F_3) = \deg g(F_1) \in 3\mathbb{N}$ , a contradiction.

Finally, assume that  $(F_1 - g(F_2, F_3), F_2, F_3)$ , where  $g \in \mathbb{C}[x, y]$ , there is an elementary reduction of  $(F_1, F_2, F_3)$ . Then  $\deg g(F_2, F_3) = 3$ . Let

$$p = \frac{d_2}{\gcd(d_2, d_3)}.$$

Since  $d_3 \notin 3\mathbb{N} + d_2\mathbb{N}$ , we obtain  $d_2 \nmid d_3$ , and hence  $p > 1$ . By Proposition 9 we have

$$\deg g(F_2, F_3) \geq q(pd_3 - d_2 - d_3 + \deg[F_1, F_3]) + rd_3,$$

where  $\deg_Y g(X, Y) = qp + r$  with  $0 \leq r < p$ . Since  $d_3 > 3$ , it follows that  $r = 0$ . Consider the case  $p \geq 3$ . Then  $pd_3 - d_2 - d_3 + \deg[F_1, F_3] \geq d_3 + \deg[F_1, F_3] > 3$ . Thus we must have  $q = 0$ . Hence  $g(X, Y) = g(X)$ , and  $3 = \deg g(F_2, F_3) = \deg g(F_2) \in d_2\mathbb{N}$ . This contradicts  $d_2 \neq 3$  (we have assumed that  $3 \nmid d_2$ ).

Consider now the case  $p = 2$ . Since  $p = 2$ , we have, for some  $n \in \mathbb{N}$ ,  $d_2 = 2n$  and  $d_3 = ns$ , where  $s \geq 3$  is odd. Since also  $d_2 > 3$ , it follows that  $n \geq 2$ . This means that  $d_3 - d_2 \geq 2$ , and  $2d_3 - d_3 - d_2 + \deg[F_1, F_3] = d_3 - d_2 + \deg[F_1, F_3] \geq 4 > 3$ . Thus, also in this case we have  $q = 0$ . As before this leads to a contradiction.  $\square$

**Corollary 44.** *The following equality holds true*

$$\begin{aligned} & \{(3, d_2, d_3) \mid 3 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)) = \\ & = \{(3, d_2, d_3) \mid 3 \leq d_2 \leq d_3, 3 \mid d_2 \text{ or } d_3 \in 3\mathbb{N} + d_2\mathbb{N}\}. \end{aligned}$$

## 6. THE CASE $(4, d_2, d_3)$

In this section we give partial description of the set

$$\{(4, d_2, d_3) \mid 4 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)).$$

This description will be given separately for four cases: (I)  $d_2, d_3$  are both even numbers, (II)  $d_2, d_3$  are both odd numbers, (III)  $d_2$  is even and  $d_3$  is odd, (IV)  $d_2$  is odd and  $d_3$  is even.

**6.1. The case  $(4, \text{even}, \text{even})$ .** This is the easiest case, summarised as follows.

**Theorem 45.** *For all even numbers  $d_3 \geq d_2 \geq 4$ ,  $(4, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .*

*Proof.* Since all numbers  $4, d_2, d_3$  are even, we have  $\gcd(4, d_2, d_3) \in \{2, 4\}$ . Thus  $\frac{4}{\gcd(4, d_2, d_3)} \leq 2$  and we can use Theorem 17.  $\square$

6.2. **The case**  $(4, \text{odd}, \text{odd})$ . In this section we give entire description of the set

$$\{(4, d_2, d_3) \mid 4 \leq d_2 \leq d_3, \quad d_2, d_3 \in 2\mathbb{N} + 1\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)).$$

We will show the following

**Theorem 46.** *Let  $d_3 \geq d_2 \geq 4$  be odd numbers. Then  $(4, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$ .*

*Proof.* Because of Proposition 15 it is enough to show that if  $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$ , then  $(4, d_2, d_3) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ . Thus, up to the end of the proof we assume that  $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$ . Since  $d_2$  is odd, we have  $\gcd(4, d_2) = 1$ , and so, by Theorem 13,

$$(27) \quad d_3 < (4 - 1)(d_2 - 1) = 3d_2 - 3.$$

By Remark 3 and Theorem 27, it is enough to show that no automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$  admits an elementary reduction.

Assume, to the contrary, that  $(F_1, F_2, F_3 - g(F_1, F_2))$ , where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of an automorphism  $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$ . Then

$$(28) \quad \deg g(F_1, F_2) = d_3.$$

By Proposition 9,

$$(29) \quad \deg g(F_1, F_2) \geq q(pd_2 - d_2 - 4 + \deg[F_1, F_2]) + rd_2,$$

where  $\deg_Y g(X, Y) = pq + r$ ,  $0 \leq r < p$  and  $p = \frac{4}{\gcd(4, d_2)} = 4$ . Since  $pd_2 - d_2 - 4 + \deg[F_1, F_2] = 3d_2 - 4 + \deg[F_1, F_2] \geq 3d_2 - 2$ , by (27), (28) and (29) we have  $q = 0$  and  $r \leq 2$ . This means that  $g(X, Y)$  is of the form

$$g(X, Y) = g_0(X) + g_1(X)Y + g_2(X)Y^2.$$

Since the sets  $4\mathbb{N}$ ,  $d_2 + 4\mathbb{N}$  and  $2d_2 + 4\mathbb{N}$  are pairwise disjoint (because  $\text{lcm}(4, d_2) = 4d_2$ ), it follows that

$$d_3 = \deg g(F_1, F_2) \in 4\mathbb{N} \cup (d_2 + 4\mathbb{N}) \cup (2d_2 + 4\mathbb{N}).$$

This contradicts  $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$ .

Now, assume that  $(F_1, F_2 - g(F_1, F_3), F_3)$ , where  $g \in \mathbb{C}[x, y]$ , is an elementary reduction of an automorphism  $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$ . Then

$$(30) \quad \deg g(F_1, F_3) = d_2.$$

But, by Proposition 9 we have

$$(31) \quad \deg g(F_1, F_3) \geq q(pd_3 - d_3 - 4 + \deg[F_1, F_3]) + rd_3,$$

where  $\deg_Y g(X, Y) = pq + r$ ,  $0 \leq r < p$  and  $p = \frac{4}{\gcd(4, d_3)} = 4$ . Since  $d_3 > d_2 > 4$ , we see that  $pd_3 - d_3 - 4 + \deg[F_1, F_3] > 2d_3 > d_2$ . Hence by (30) and (31),  $q = r = 0$ . This means that  $g(X, Y) = g(X)$  and so  $d_2 = \deg g(F_1, F_3) = \deg g(F_1) \in 4\mathbb{N}$ . This contradicts the assumption that  $d_2$  is an odd number.

Finally, assume that  $(F_1 - g(F_2, F_3), F_2, F_3)$ , where  $g \in \mathbb{C}[x, y]$ , is an elementary reduction of an automorphism  $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$ . Then

$$(32) \quad \deg g(F_2, F_3) = 4.$$

By Proposition 9,

$$(33) \quad \deg g(F_1, F_3) \geq q(pd_3 - d_3 - d_2 + \deg[F_2, F_3]) + rd_3,$$

where  $\deg_Y g(X, Y) = pq + r$ ,  $0 \leq r < p$  and  $p = \frac{d_2}{\gcd(d_2, d_3)}$ . Since  $d_3 > 4$ , by (32) and (33) we have  $r = 0$ . Since also  $2 \nmid d_2$  and  $d_2 \nmid d_3$  (because  $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$ ), we conclude that  $p = \frac{d_2}{\gcd(d_2, d_3)} \geq 3$  and  $pd_3 - d_3 - d_2 + \deg[F_2, F_3] > d_3 > 4$ . Thus  $q = 0$ . Then we obtain a contradiction as in the previous case.  $\square$

**Corollary 47.** *The following equality holds true:*

$$\begin{aligned} & \{(4, d_2, d_3) : 4 \leq d_2 \leq d_3, d_2, d_3 \in 2\mathbb{N} + 1\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)) = \\ & = \{(4, d_2, d_3) : 4 \leq d_2 \leq d_3, d_2, d_3 \in 2\mathbb{N} + 1, d_3 \in 4\mathbb{N} + d_2\mathbb{N}\}. \end{aligned}$$

**6.3. The case (4, even, odd).** We start this subsection with the following two examples (or rather two series of examples).

**Example 2.** *Since*

$$\begin{aligned} (X + Z^4)^3 &= Z^{12} + 3XZ^8 + 3X^2Z^4 + X^3, \\ (Y + Z^6)^2 &= Z^{12} + 2YZ^6 + Y^2, \end{aligned}$$

*we see that*

$$\deg[(Y + Z^6)^2 - (X + Z^4)^3] = 9.$$

*Thus, for any  $k \in \mathbb{N}$ ,*

$$\deg[(Y + Z^6)^2 - (X + Z^4)^3](X + Z^4)^k = 9 + 4k.$$

*This means that*

$$\text{mdeg}(F_2 \circ F_1) = (4, 6, 9 + 4k),$$

*where*

$$\begin{aligned} F_1(x, y, z) &= (x + z^4, y + z^6, z), \\ F_2(u, v, w) &= (u, v, w + (v^2 - u^3)u^k). \end{aligned}$$

**Example 3.** *Since*

$$\begin{aligned} (X + Z^4)^3 &= Z^{12} + 3XZ^8 + 3X^2Z^4 + X^3, \\ (Y + \tfrac{3}{2}XZ^2 + Z^6)^2 &= Z^{12} + 3XZ^8 + 2YZ^6 + \tfrac{9}{4}X^2Z^4 + 3YXZ^2 + Y^2, \end{aligned}$$

*it follows that*

$$\deg[(Y + \tfrac{3}{2}XZ^2 + Z^6)^2 - (X + Z^4)^3] = 7,$$

*and*

$$\deg[(Y + \tfrac{3}{2}XZ^2 + Z^6)^2 - (X + Z^4)^3](X + Z^4)^k = 7 + 4k.$$

*Thus we have*

$$\text{mdeg}(F_2 \circ F_1) = (4, 6, 7 + 4k),$$

*where*

$$\begin{aligned} F_1(x, y, z) &= (x + z^4, y + \tfrac{3}{2}xz^2 + z^6, z), \\ F_2(u, v, w) &= (u, v, w + (v^2 - u^3)u^k). \end{aligned}$$

Combining above examples and Theorem 45 we obtain the following

**Proposition 48.** *For any integer  $d_3 \geq 6$  we have  $(4, 6, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .*

In the same manner one can prove the following

**Proposition 49.** *For any integer  $d_3 \geq 10$  we have  $(4, 10, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .*



Using Corollary 16 we obtain

**Proposition 50.** *For  $k = 1, 2, \dots$  and an integer  $d_3 \geq 4k$  we have  $(4, 4k, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .*

The next proposition gives partial information about multidegrees of the form  $(4, 4k + 2, d_3)$ , where  $k = 3, 4, \dots$  and  $d_3 \geq 4k + 2$ .

**Proposition 51.** *For integers  $k \geq 3$  and  $d_3 \geq 5k + 1$  we have  $(4, 4k + 2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .*

*Proof.* Let us notice that

$$(X + Z^4)^{2k+1} = \sum_{l=0}^{2k+1} \binom{2k+1}{l} X^l Z^{8k+4-4l}$$

and

$$\begin{aligned} \left( Y + Z^r + \sum_{l=0}^k a_l X^l Z^{4k+2-4l} \right)^2 &= Y^2 + 2YZ^r + Z^{2r} + 2Y \sum_{l=0}^k a_l X^l Z^{4k+2-4l} \\ &\quad + 2Z^r \sum_{l=0}^k a_l X^l Z^{4k+2-4l} \\ &\quad + \sum_{s=0}^{2k} \left( \sum_{l+m=s, l,m \in \{0, \dots, k\}} a_l a_m \right) X^s Z^{8k+4-4s}. \end{aligned}$$

We will consider the cases  $r = k - 1, k, k + 1$  and  $k + 2$ . Thus we have:

$$\begin{aligned} \deg 2YZ^r &\leq k + 3 < 5k + 1, \\ \deg Z^{2r} &\leq 2k + 4 < 5k + 1, \\ \deg 2Y \sum_{l=0}^k a_l X^l Z^{4k+2-4l} &\leq 4k + 3 < 5k + 1, \\ \deg 2Z^r \sum_{l=0}^k a_l X^l Z^{4k+2-4l} &\leq 5k - 2 < 5k + 1. \end{aligned}$$

This means that the only summands of the polynomial

$$(34) \quad (X + Z^4)^{2k+1} - \left( Y + Z^r + \sum_{l=0}^k a_l X^l Z^{4k+2-4l} \right)^2$$

of degree greater or equal to  $5k + 1$  are:

$$\begin{aligned}
& (1 - a_0^2) Z^{8k+4}, \\
& \left[ \binom{2k+1}{1} - 2a_0a_1 \right] X Z^{8k}, \\
& \left[ \binom{2k+1}{2} - (2a_0a_2 + a_1^2) \right] X^2 Z^{8k-4}, \\
& \vdots \\
& \left[ \binom{2k+1}{k} - (a_0a_k + a_1a_{k-1} + \cdots + a_{k-1}a_1 + a_ka_0) \right] X^k Z^{4k+4}, \\
& 2a_0z^{4k+2+r}
\end{aligned}$$

and (only in the case  $r = k + 2$ )

$$2a_1XZ^{4k-2+r}.$$

Since we can recursively solve the following system of equations (notice that we can take  $a_0 = 1$ )

$$\begin{aligned}
1 - a_0^2 &= 0, \\
\binom{2k+1}{1} - 2a_0a_1 &= 0, \\
\binom{2k+1}{2} - (2a_0a_2 + a_1^2) &= 0, \\
&\vdots \\
\binom{2k+1}{k} - (a_0a_k + a_1a_{k-1} + \cdots + a_{k-1}a_1 + a_ka_0) &= 0,
\end{aligned}$$

it follows that we can choose coefficients  $a_0, a_1, \dots, a_k$  such that the degree of the polynomial (34) is equal to

$$\deg(2a_0Z^{4k+2+r}) = 4k + 2 + r.$$

Taking  $r = k - 1, k, k + 1$  and  $k + 2$  we obtain polynomials of degree equal to  $5k + 1, 5k + 2, 5k + 3$  and  $5k + 4$ , respectively.

Now, it is easy to see that taking

$$F(x, y, z) = \left( x + z^4, y + z^r + \sum_{l=0}^k a_l x^l z^{4k+2-4l}, z \right)$$

and

$$G(u, v, w) = (u, v, w + (u^{4k+1} - v^2)u^q),$$

where  $q = 0, 1, \dots$ , we obtain that

$$\text{mdeg}(G \circ F) = (4, 4k + 2, 4k + 2 + r + 4q).$$

Since for any  $d_3 \geq 5k + 1$  we can find  $r \in \{k - 1, k, k + 1, k + 2\}$  and  $q \in \mathbb{N}$  such that  $4k + 2 + r + 4q = d_3$ , the result follows.  $\square$

**6.4. The case  $(4, \text{odd}, \text{even})$ .** In this subsection we give almost complete description of the set

$$\{(4, d_2, d_3) \mid 4 \leq d_2 \leq d_3, \quad d_2 \in 2\mathbb{N} + 1, d_3 \in 2\mathbb{N}\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)).$$

Namely we have the following result.

**Theorem 52.** *If  $d_2 \geq 5$  is an odd number and  $d_3 \geq d_2$  is an even number such that  $d_3 - d_2 \neq 1$ , then  $(4, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$ .*

*Proof.* If  $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$ , then by Proposition 15,  $(4, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ . Thus up to the end of the proof we will assume that  $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$ . Our purpose is to show that there is no tame automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$ . Since  $d_2$  is an odd number, by Remark 3 and Theorem 27 it is enough to show that all automorphisms  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$  do not admit an elementary reduction. Thus the rest of the proof is the inspection of the three cases of elementary reducibility.

Assume that  $(F_1, F_2, F_3 - g(F_1, F_2))$ , where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of an automorphism  $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$ . Thus

$$\deg g(F_1, F_2) = d_3,$$

and by Proposition 9,

$$\deg g(F_1, F_2) \geq q(pd_2 - d_2 - 4 + \deg[F_1, F_2]) + rd_2,$$

where  $\deg_Y g(X, Y) = pq + r$ ,  $0 \leq r < p$  and  $p = \frac{4}{\gcd(4, d_2)} = 4$ . Since  $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$  and  $\gcd(4, d_2) = 1$ , we have (similarly as in the proof of Theorem 46)

$$(35) \quad d_3 < 3d_2 - 3.$$

Thus we can repeat the arguments from the corresponding case in the proof of Theorem 46 to obtain a contradiction.

Now, assume that  $(F_1, F_2 - g(F_1, F_3), F_3)$ , for some  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of an automorphism  $F = (F_1, F_2, F_3)$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (4, d_2, d_3)$ . Then

$$(36) \quad \deg g(F_1, F_3) = d_2,$$

and by Proposition 9,

$$(37) \quad \deg g(F_1, F_3) \geq q(pd_3 - d_3 - 4 + \deg[F_1, F_2]) + rd_3,$$

where  $\deg_Y g(X, Y) = pq + r$ ,  $0 \leq r < p$  and  $p = \frac{4}{\gcd(4, d_3)} = 2$  (because  $d_3$  is an even number and  $d_3 \notin 4\mathbb{N} + d_2\mathbb{N}$ ). Thus  $pd_3 - d_3 - 4 + \deg[F_1, F_2] \geq d_3 - 2$ . But by the assumptions  $d_3 - d_2 \geq 0$  is an odd number different from 1. So  $d_2 \leq d_3 - 3$ , and then  $pd_2 - d_2 - 4 + \deg[F_1, F_2] > d_2$ . Consequently, by (36) and (37),  $q = 0$ . Since also  $r = 0$  (because  $d_3 > d_2$ ), we see that  $g(X, Y) = g(X)$ , and so

$$d_2 = \deg g(F_1, F_3) = \deg g(F_1) \in 4\mathbb{N}.$$

This contradicts the assumption that  $d_2$  is an odd number.

In the last case we can repeat the arguments from the corresponding case in the proof of Theorem 46  $\square$

**Corollary 53.** *If  $d_2 \geq 5$  is an odd number such that  $d_2 \equiv 3 \pmod{4}$ , and  $d_3 \geq d_2$  is an even number, then  $(4, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$ .*

*Proof.* Let us notice, that if  $d_3 - d_2 = 1$ , then  $4|d_3$ . Thus  $d_3 \in 4\mathbb{N} + d_2\mathbb{N}$  and by Proposition 15  $(4, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ . In the case  $d_3 - d_2 > 1$ , we can use Theorem 52.  $\square$

By the above corollary, we know that to complete the picture of the set

$$\{(4, d_2, d_3) \mid 4 \leq d_2 \leq d_3, \quad d_2 \in 2\mathbb{N} + 1, d_3 \in 2\mathbb{N}\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$$

it is enough to consider the triples of the form

$$(4, 4k + 1, 4k + 2) \quad \text{for } k = 1, 2, \dots$$

Moreover, using the arguments from the proof of Theorem 52, one can show the following

**Proposition 54.** *Let  $k \in \mathbb{N}^*$ . If there exists a tame automorphism  $\tilde{F}$  of  $\mathbb{C}^3$  with  $\text{mdeg } \tilde{F} = (4, 4k + 1, 4k + 2)$ , then there is also a tame automorphism  $F = (F_1, F_2, F_3)$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (4, 4k + 1, 4k + 2)$  that admits an elementary reduction  $(F_1, F_2 - g(F_1, F_3), F_3)$ , for some  $g \in \mathbb{C}[X, Y]$ . Moreover, for such automorphism  $F$  we have  $\deg[F_1, F_3] \leq 3$ .*

Using arguments from the proof of Theorem 57 one can also show that  $\deg[F_1, F_3] = 3$  when  $k < 25$ .

## 7. THE CASE $(p, d_2, d_3)$ AND $(5, d_2, d_3)$

**7.1. The general case.** Now we make, in some sense, a generalization of the results of section 'The case  $(3, d_2, d_3)$ '. This generalization is not complete, because in the presented picture there are some holes. The first, general result is the following

**Theorem 55.** *Let  $2 \leq p \leq d_2 \leq d_3$  be a sequence of positive integers, and let  $p$  be a prime number. If*

$$(1) \quad \frac{d_3}{d_2} \neq \frac{3}{2} \text{ or}$$

$$(2) \quad \frac{d_3}{d_2} = \frac{3}{2} \text{ and } \frac{d_2}{2} > p - 2,$$

*then  $(p, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $p|d_2$  or  $d_3 \in p\mathbb{N} + d_2\mathbb{N}$ .*

*Proof.* By Corollary 15, if  $p|d_2$  or  $d_3 \in p\mathbb{N} + d_2\mathbb{N}$ , then there exists a tame automorphism  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\text{mdeg } F = (p, d_2, d_3)$ . Thus in order to prove Theorem 55 it is enough to show that if  $p \nmid d_2$  and  $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$  and (1) or (2) holds, then  $(p, d_2, d_3) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .

So let us assume that  $p \nmid d_2, d_3 \notin p\mathbb{N} + d_2\mathbb{N}$  and (1) or (2) hold. In particular  $p < d_2 < d_3$ . By Theorems 43 and 16, we can assume that  $p > 3$ . Indeed, for  $p = 2$ , by Corollary 16 we have  $(2, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  for all integers  $2 \leq d_2 \leq d_3$ . Also the condition  $2|d_2$  or  $d_3 \in 2\mathbb{N} + d_2\mathbb{N}$  is satisfied for all integers  $2 \leq d_2 \leq d_3$ . For  $p = 3$  we simply use Theorem 43. So up to the end of the proof we will assume that  $p > 3$ . Thus by Theorem 27 it is enough to show that no automorphism  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (p, d_2, d_3)$  admits an elementary reduction (notice that  $3 \nmid p$ ).

Assume, to the contrary, that there exists an automorphism  $F = (F_1, F_2, F_3)$  with  $\text{mdeg } F = (p, d_2, d_3)$  that admits an elementary reduction. Since  $p \nmid d_2$ , we have  $\gcd(p, d_2) = 1$ . So by Theorem 13 we have  $k \in p\mathbb{N} + d_2\mathbb{N}$  for all  $k \geq (p-1)(d_2-1) = pd_2 - d_2 - p + 1$ . Thus

$$(38) \quad d_3 < pd_2 - d_2 - p + 1,$$

since  $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$ .

Assume that

$$(F_1, F_2, F_3 - g(F_1, F_2)),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Hence we have  $\deg g(F_1, F_2) = \deg F_3 = d_3$ . Since  $\gcd(p, d_2) = 1$ , we see that  $\frac{p}{\gcd(p, d_2)} = p$ , and so by Proposition 9,

$$\deg g(F_1, F_2) \geq q(pd_2 - d_2 - p + \deg[F_1, F_2]) + rd_2,$$

where  $\deg_Y g(X, Y) = pq + r$  with  $0 \leq r < p$ . Since  $F_1, F_2$  are algebraically independent,  $\deg[F_1, F_2] \geq 2$  and  $pd_2 - d_2 - p + \deg[F_1, F_2] \geq pd_2 - d_2 - p + 2$ . Then by (38) follows that  $q = 0$ . Thus

$$g(X, Y) = \sum_{i=0}^{p-1} g_i(X)Y^i.$$

Since  $\text{lcm}(p, d_2) = pd_2$ , the sets

$$p\mathbb{N}, d_2 + p\mathbb{N}, \dots, (p-1)d_2 + p\mathbb{N}$$

are pairwise disjoint. So

$$\deg \left( \sum_{i=0}^{p-1} g_i(F_1)F_2^i \right) = \max_{i=0, \dots, p-1} (\deg F_1 \deg g_i + i \deg F_2)$$

and

$$\begin{aligned} d_3 &= \deg g(F_1, F_2) \\ &= \deg \left( \sum_{i=0}^{p-1} g_i(F_1)F_2^i \right) \in \bigcup_{r=0}^{p-1} (rd_2 + p\mathbb{N}) \subset p\mathbb{N} + d_2\mathbb{N}, \end{aligned}$$

a contradiction.

Now assume that

$$(F_1, F_2 - g(F_1, F_3), F_3),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Since  $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$ , we have  $p \nmid d_3$  and  $\gcd(p, d_3) = 1$ . Hence by Proposition 9,

$$\deg g(F_1, F_3) \geq q(pd_3 - d_3 - p + \deg[F_1, F_3]) + rd_3,$$

where  $\deg_Y g(X, Y) = qp + r$  with  $0 \leq r < p$ . Since  $pd_3 - d_3 - p + \deg[F_1, F_3] \geq pd_3 - 2d_3 \geq 3d_3 > d_2$  and since we want to have  $\deg g(F_1, F_3) = p_2$ , we conclude that  $q = r = 0$ . This means that  $g(X, Y) = g(X)$ , and so

$$d_2 = \deg g(F_1, F_2) = \deg g(F_1) \in p\mathbb{N} \subset p\mathbb{N} + d_2\mathbb{N},$$

a contradiction.

Finally, assume that

$$(F_1 - g(F_2, F_3), F_2, F_3),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Thus we have  $\deg g(F_2, F_3) = p$ . Let

$$\tilde{p} = \frac{d_2}{\gcd(d_2, d_3)}.$$

Since  $d_3 \notin p\mathbb{N} + d_2\mathbb{N}$ , we see that  $d_2 \nmid d_3$ , and so  $\tilde{p} > 1$ . By Proposition 9,

$$\deg g(F_2, F_3) \geq q(\tilde{p}d_3 - d_2 - d_3 + \deg[F_1, F_3]) + rd_3,$$

where  $\deg_Y g(X, Y) = q\tilde{p} + r$  with  $0 \leq r < \tilde{p}$ . Since  $d_3 > p$  (because  $d_3 > d_2 > p$ ), we see that  $r = 0$ . Consider the case  $\tilde{p} \geq 3$ . Then  $\tilde{p}d_3 - d_2 - d_3 + \deg[F_1, F_3] \geq d_3 + \deg[F_1, F_3] > p$ . Thus we must have  $q = 0$ . Hence  $g(X, Y) = g(X)$  and

$$p = \deg g(F_2, F_3) = \deg g(F_2) \in d_2\mathbb{N}.$$

This contradicts  $d_2 \neq p$  (we have assumed that  $p \nmid d_2$ ).

Now, consider the case  $\tilde{p} = 2$ . Since  $\tilde{p} = 2$ , we have, for some  $n \in \mathbb{N}^*$ ,  $d_2 = 2n$  and  $d_3 = ns$ , where  $s \geq 3$  is odd. Consider first the case  $s > 3$ . Then

$$\begin{aligned} 2d_3 - d_3 - d_2 + \deg[F_1, F_3] &= d_3 - d_2 + \deg[F_1, F_3] \\ &= (s-2)n + \deg[F_1, F_3] > d_2 > p. \end{aligned}$$

Thus we have  $q = 0$ . As before this leads to a contradiction.

Now, consider the case  $s = 3$ . This is the case when we use the second statement of the assumption (2). Since  $d_2 = 2n$  and  $d_3 = 3n$ , we see that  $\frac{d_3}{d_2} = \frac{3}{2}$ . Hence (1) is not satisfied. Thus, the assumption (2) is satisfied and so  $n = \frac{d_2}{2} > p - 2$ . Hence

$$\begin{aligned} 2d_3 - d_3 - d_2 + \deg[F_1, F_3] &= d_3 - d_2 + \deg[F_1, F_3] \geq \\ &\geq n + 2 > p. \end{aligned}$$

So, also in this case we have  $q = 0$ . As before this leads to a contradiction.  $\square$

For small prime numbers  $p$  the above theorem gives, for example, the following results.

**Corollary 56.** (a) If  $(5, d_2, d_3) \neq (5, 6, 9)$  and  $5 \leq d_2 \leq d_3$ , then  $(5, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $5|d_2$  or  $d_3 \in 5\mathbb{N} + d_2\mathbb{N}$ .  
(b) If  $(7, d_2, d_3) \notin \{(7, 8, 12), (7, 10, 15)\}$  and  $7 \leq d_2 \leq d_3$ , then  $(7, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $7|d_2$  or  $d_3 \in 7\mathbb{N} + d_2\mathbb{N}$ .  
(c) If  $(11, d_2, d_3) \notin \{(11, 12, 18), (11, 14, 21), (11, 16, 24), (11, 18, 27)\}$  and  $11 \leq d_2 \leq d_3$ , then  $(11, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $11|d_2$  or  $d_3 \in 11\mathbb{N} + d_2\mathbb{N}$ .  
(d) If  $(13, d_2, d_3) \notin \{(13, 14, 21), (13, 16, 24), (13, 18, 27), (13, 20, 30), (13, 22, 33)\}$  and  $13 \leq d_2 \leq d_3$ , then  $(13, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  if and only if  $13|d_2$  or  $d_3 \in 13\mathbb{N} + d_2\mathbb{N}$ .

*Proof.* One can easily check that, for example, for  $p = 11$  the only triples of the form  $(11, d_2, d_3)$ , with  $11 \leq d_2 \leq d_3$ , that does not satisfy neither condition (1) nor condition (2) of the above theorem are  $(11, 12, 18), (11, 14, 21), (11, 16, 24)$  and  $(11, 18, 27)$ .  $\square$

The point (a) of the above corollary says that we have almost complete description of the set

$$(39) \quad \{(5, d_2, d_3) : 5 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)).$$

The only one thing that we do not know is whether  $(5, 6, 9)$  is an element of this set. One can, of course, notice that  $9 \notin 5\mathbb{N} + 6\mathbb{N}$ . In the next section we show that  $(5, 6, 9) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ , and so we obtain the complete description of the set (39).

**7.2. Tame automorphism of  $\mathbb{C}^3$  with multidegree equal  $(5, 6, 9)$  and the Jacobian Conjecture.** Our main purpose in this section is to prove the following result.

**Theorem 57.** *There is no tame automorphism of  $\mathbb{C}^3$  with multidegree equal to  $(5, 6, 9)$ .*

Before we give the proof of the above theorem we recall some positive results about the Jacobian Conjecture in dimension two. In the proof of the theorem we can use one of such results but for the completeness we recall a little bit more.

The first one is the following result due to Magnus [30].

**Theorem 58.** *(Magnus, see also [7], Thm. 10.2.24) Let  $F = (P, Q)$  be a Keller map (i.e. such that  $\text{Jac} F = 1$ ). If  $\gcd(\deg P, \deg Q) = 1$  then  $F$  is invertible and  $\deg P = 1$  or  $\deg Q = 1$ .*

The next, also due to Magnus, is the following corollary of the above theorem.

**Corollary 59.** *(Magnus, see e.g. [7]) If  $F = (P, Q)$  is a Keller map and  $\deg P$  or  $\deg Q$  is a prime number, then  $F$  is invertible.*

Later Applegate, Onishi and Nagata improved the result of Magnus.

**Theorem 60.** *(Applegate, Onishi, Nagata, see e.g. [3, 4] or [7]) Let  $F = (P, Q)$  be a Keller map and  $d = \gcd(\deg P, \deg Q)$ . If  $d \leq 8$  or  $d$  is a prime number, then  $F$  is invertible.*

The last result that we recall here is the following one due to Moh [33].

**Theorem 61.** *(see also [7]) Let  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be a Keller map with  $\deg F \leq 101$ . Then  $F$  is invertible.*

Now we can give the proof of Theorem 57.

*Proof.* By Theorem 27, it is enough to show that no (hypothetical) automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (5, 6, 9)$  admits an elementary reduction. Moreover, it is enough to show this for automorphisms  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $F(0, 0, 0) = (0, 0, 0)$ .

Assume, to the contrary, that there is an automorphism  $F = (F_1, F_2, F_3) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  with  $\text{mdeg } F = (5, 6, 9)$  that admits an elementary reduction.

Assume that

$$(F_1, F_2, F_3 - g(F_1, F_2)),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Then

$$(40) \quad \deg g(F_1, F_2) = \deg F_3 = 9.$$

By Proposition 9,

$$(41) \quad \deg g(F_1, F_2) \geq q(5 \cdot 6 - 6 - 5 + \deg[F_1, F_2]) + 6r,$$

where  $\deg_Y g(X, Y) = 5q + r$ , with  $0 \leq r < 5$ . Since  $5 \cdot 6 - 6 - 5 + \deg[F_1, F_2] \geq 19 + \deg[F_1, F_2] > 9$ , by (40) and (41) we have  $q = 0$ . Also by (40) and (41) we have  $r < 2$ . Thus  $g(X, Y) = g_0(X) + Yg_0(X)$ , and since  $5\mathbb{N} \cap (6 + 5\mathbb{N}) = \emptyset$ , it follows that

$$9 = \deg g(F_1, F_2) \in 5\mathbb{N} \cup (6 + 5\mathbb{N}),$$

a contradiction.

Now, assume that

$$(F_1, F_2 - g(F_1, F_3), F_3),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . Then

$$(42) \quad \deg g(F_1, F_3) = \deg F_2 = 6.$$

By Proposition 9

$$(43) \quad \deg g(F_1, F_3) \geq q(5 \cdot 9 - 9 - 5 + \deg[F_1, F_3]) + 9r,$$

where  $\deg_Y g(X, Y) = 5q + r$ , with  $0 \leq r < 5$ . Since  $5 \cdot 9 - 9 - 5 + \deg[F_1, F_3] \geq 31 + \deg[F_1, F_3] > 6$ , we have  $q = r = 0$ . This means that  $g(X, Y) = g(X)$ , and so

$$\deg g(F_1, F_2) = \deg g(F_1) \in 5\mathbb{N},$$

a contradiction.

Finally, assume that

$$(F_1 - g(F_2, F_3), F_2, F_3),$$

where  $g \in \mathbb{C}[X, Y]$ , is an elementary reduction of  $(F_1, F_2, F_3)$ . By Theorem 27, we can also assume that  $F(0, 0, 0) = (0, 0, 0)$ . We have

$$(44) \quad \deg g(F_2, F_3) = \deg F_1 = 5$$

and by Proposition 9,

$$(45) \quad \deg g(F_2, F_3) \geq q(p \cdot 9 - 9 - 6 + \deg[F_2, F_3]) + 9r,$$

where  $\deg_Y g(X, Y) = qp + r$ , with  $0 \leq r < p$  and  $p = \frac{6}{\gcd(6, 9)} = 2$ . By (44) and (45),  $r = 0$ .

Consider the case  $\deg[F_2, F_3] > 2$ . Then  $p \cdot 9 - 9 - 6 + \deg[F_2, F_3] = 3 + \deg[F_2, F_3] > 5$ , and then by (44) and (45) we see that  $q = 0$ . Thus in this case, we have  $g(X, Y) = g(X)$ , and so  $\deg g(F_2, F_3) = \deg g(F_2) \in 6\mathbb{N}$ . This contradicts (44).

Now, consider the case  $\deg[F_2, F_3] = 2$  (since  $F_2, F_3$  are algebraically independent, we have  $\deg[F_2, F_3] \geq 2$ ). Let  $L$  be the linear part of the automorphism  $F$ . Since  $F(0, 0, 0) = (0, 0, 0)$ , the linear part of  $F \circ L^{-1}$  is the identity map  $\text{id}_{\mathbb{C}^3}$ . Thus

$$(46) \quad \begin{aligned} F_2 \circ L^{-1} &= X_2 + \text{higher degree summands}, \\ F_3 \circ L^{-1} &= X_3 + \text{higher degree summands}. \end{aligned}$$

Since, by Lemma 10,

$$\deg[F_2 \circ L^{-1}, F_3 \circ L^{-1}] = \deg[F_2, F_3] = 2,$$

it follows, by Lemma 32, that

$$F_2 \circ L^{-1}, F_3 \circ L^{-1} \in \mathbb{C}[X_2, X_3].$$

But  $\deg[F_2 \circ L^{-1}, F_3 \circ L^{-1}] = 2$  means that

$$\text{Jac}(F_2 \circ L^{-1}, F_3 \circ L^{-1}) \in \mathbb{C}^*$$

(of course we consider here  $F_2 \circ L^{-1}, F_3 \circ L^{-1}$  as functions of two variables  $X_2, X_3$ ). By Lemma 29 we have  $\deg(F_2 \circ L^{-1}) = 6$ ,  $\deg(F_3 \circ L^{-1}) = 9$ . Then, by Theorem 61, the map  $(F_2 \circ L^{-1}, F_3 \circ L^{-1}) : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  is an automorphism. But  $6 \nmid 9$  contradict Jung - van der Kulk theorem (see Theorem 3 and Corollary 2).  $\square$

By Theorem 57 and Corollary 56 (a) we obtain the following result.



**Corollary 62.** *The following equality holds true*

$$\begin{aligned} & \{(5, d_2, d_3) : 5 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3)) = \\ & = \{(5, d_2, d_3) : 5 \leq d_2 \leq d_3, 5|d_2 \text{ or } d_3 \in 5\mathbb{N} + d_2\mathbb{N}\}. \end{aligned}$$

**7.3. The case  $(p, 2(p-2), 3(p-2))$ .** In the same manner as we proved Theorem 57 one can show the following

**Theorem 63.** *Let  $p \geq 5$  be a prime number such that  $p \leq 35$ . Then  $(p, 2(p-2), 3(p-3)) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .*

*Proof.* Since  $3(p-2) \leq 101$ , it follows that one can use Theorem 61 and repeat the arguments from the proof of Theorem 57.  $\square$

By the above theorem and Corollary 56 we obtain the following result.

**Corollary 64.** *The following equality holds true*

$$\begin{aligned} & [\{(7, d_2, d_3) : 7 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))] \setminus \{(7, 8, 12)\} = \\ & = \{(7, d_2, d_3) : 7 \leq d_2 \leq d_3, 7|d_2 \text{ or } d_3 \in 7\mathbb{N} + d_2\mathbb{N}\}. \end{aligned}$$

The above corollary means that to obtain the complete description of the set  $\{(7, d_2, d_3) : 7 \leq d_2 \leq d_3\} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$  we "only" need to know whether  $(7, 8, 12) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .

At the end of this subsection notice the following result.

**Theorem 65.** *The Jacobian Conjecture for dimension two implies that for a prime numbers  $p \geq 5$  we have  $(p, 2(p-2), 3(p-2)) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ .*

*Proof.* If we assume that the Jacobian Conjecture for dimension two holds true, then one can repeat the arguments from the proof of Theorem 57.  $\square$

**Corollary 66.** *If there is a tame automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (p, 2(p-2), 3(p-2))$ , where  $p > 35$  is a prime number, then the Jacobian Conjecture for dimension two is false.*

*Proof.* This is a consequence of Theorem 63 and Theorem 65.  $\square$

In particular we have the following

**Theorem 67.** *If there is a tame automorphism  $F$  of  $\mathbb{C}^3$  with  $\text{mdeg } F = (37, 70, 105)$ , then the two-dimensional Jacobian Conjecture is false.*

## 8. FINITENESS RESULTS

Let us consider the set

$$T_{a,b}^{(n)} = \{(d_1, \dots, d_n) \in (\mathbb{N}^*)^n : d_1 \leq \dots \leq d_n, d_1 = a, d_2 = b\} \setminus \text{mdeg}(\text{Tame}(\mathbb{C}^n)).$$

Of course, by Jung-Van der Kulk result,  $T_{a,b}^{(2)} = \{(a, b)\}$  if  $a \nmid b$ , and  $T_{a,b}^{(2)} = \emptyset$  if  $a|b$ . Thus  $\#T_{a,b}^{(2)} \leq 1 < +\infty$  for all  $1 \leq a \leq b$ . We will show that also for  $n \geq 3$  the set  $T_{a,b}^{(n)}$  is finite. For  $n = 3$  this result is due to Zygadlo [52].

**Theorem 68.** *For all integers  $1 \leq a \leq b$  the set  $T_{a,b}^{(3)}$  is finite. Moreover the following inclusion is true*

$$T_{a,b}^{(3)} \subset \{(a, b, d_3) : d_3 < \text{lcm}(a, b) - r\},$$

where  $r = \min\{b - 1, (a - 1)(\lfloor \frac{b}{a} \rfloor + 1)\}$ .

The original proof of the above theorem due to Zygadlo can be found in [52], but we give here another, simpler proof. It is based on the proof of Proposition 51, but there are also similarities to the proof in [52].

*Proof.* First of all notice that without loss of generality we can assume that  $1 < a < b$ . Indeed, if  $a = 1$  or  $a = b$ , then by Proposition 15 we have  $T_{a,b}^{(3)} = \emptyset$ . Thus up to the end of the proof we assume that  $1 < a < b$ .

Let  $d = \gcd(a, b)$ . Then  $a = d\tilde{a}$ ,  $b = d\tilde{b}$ , where  $\tilde{a}, \tilde{b} \in \mathbb{N}^*$  are coprime numbers. We have  $\text{lcm}(a, b) = \frac{ab}{\gcd(a, b)} = \tilde{a}\tilde{b} = b\tilde{a}$ . Let us notice that

$$(47) \quad (X + Z^a)^{\tilde{b}} = \sum_{l=0}^{\tilde{b}} \binom{\tilde{b}}{l} X^l Z^{a\tilde{b}-la}$$

and

$$(48) \quad \left( Y + Z^p + \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{\tilde{a}} =$$

$$= \sum_{s_1+s_2=\tilde{a}, s_1>0} (Y + Z^p)^{s_1} \left( \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{s_2} + \left( \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{\tilde{a}}.$$

If we take  $p < b$ , then

$$\deg \left[ \sum_{s_1+s_2=\tilde{a}, s_1>0} (Y + Z^p)^{s_1} \left( \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{s_2} \right] \leq p + b(\tilde{a} - 1),$$

and since  $Z^{p+b(\tilde{a}-1)}$  can be obtained in the above polynomial only in one way, that we actually have (provided that  $a_0 \neq 0$ ):

$$(49) \quad \deg \left[ \sum_{s_1+s_2=\tilde{a}, s_1>0} (Y + Z^p)^{s_1} \left( \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{s_2} \right] = p + b(\tilde{a} - 1).$$

In the sequel, we will take  $p \in \{1, \dots, b-1\}$  such that  $p+b(\tilde{a}-1) = \text{lcm}(a, b) - r, \dots, \text{lcm}(a, b) - r + (a-1)$ . This is possible, because  $b(\tilde{a}-1) + 1 \leq \text{lcm}(a, b) - r$  and  $\text{lcm}(a, b) - r + (a-1) < \text{lcm}(a, b) = b\tilde{a}$ .

Now, using (47), (48) and (49) we obtain that the summands of degree greater than  $p+b(\tilde{a}-1)$  in the polynomial

$$(X + Z^a)^{\tilde{b}} - \left( Y + Z^p + \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{\tilde{a}}$$

are

$$\begin{aligned} & (1 - a_0^{\tilde{a}}) Z^{a\tilde{b}}, \\ & \left[ \binom{\tilde{b}}{1} - \binom{\tilde{a}}{1} a_0^{\tilde{a}-1} a_1 \right] X Z^{a(\tilde{b}-1)}, \\ & \left[ \binom{\tilde{b}}{2} - \binom{\tilde{a}}{2} a_0^{\tilde{a}-2} a_1^2 - \binom{\tilde{a}}{1} a_0^{\tilde{a}-1} a_2 \right] X^2 Z^{a(\tilde{b}-2)}, \end{aligned}$$

and for  $k = 3, \dots, \lfloor \frac{b}{a} \rfloor$

$$\left[ \binom{\tilde{b}}{k} - \left( \sum_{l_1+\dots+l_{\tilde{a}}=k, l_i < k} a_{l_1} \cdots a_{l_{\tilde{a}}} \right) - \binom{\tilde{a}}{1} a_0^{\tilde{a}-1} a_k \right] X^k Z^{a(\tilde{b}-k)}.$$

Thus we can recursively choose coefficients  $a_0, \dots, a_{\lfloor \frac{b}{a} \rfloor}$  such that all expressions in the brackets above are equal to zero. Since also in the polynomial

$$(X + Z^a)^{\tilde{b}} - \left( \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{\tilde{a}}$$

there is no summands belonging to  $\mathbb{C}[Z] \setminus \mathbb{C}$  (provided that  $a_0 = 1$ ), then

$$\deg \left[ (X + Z^a)^{\tilde{b}} - \left( Y + Z^p + \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l X^l Z^{b-la} \right)^{\tilde{a}} \right] = p + b(\tilde{a}-1).$$

Now, let  $d_3 \geq \text{lcm}(a, b) - r$  be arbitrary. Then there are  $p \in \{1, \dots, b-1\}$  and  $q \in \mathbb{N}$  such that  $p+b(\tilde{a}-1) \in \{\text{lcm}(a, b) - r, \dots, \text{lcm}(a, b) - r + (a-1)\}$  and  $d_3 = p + b(\tilde{a}-1) + qa$ . By the above considerations we obtain that

$$\text{mdeg}(G \circ F) = (a, b, d_3),$$

where

$$F(x, y, z) = \left( x + z^a, y + z^p + \sum_{l=0}^{\lfloor \frac{b}{a} \rfloor} a_l x^l z^{b-la}, z \right)$$

and

$$G(u, v, w) = \left( u, v, w + \left( u^{\tilde{b}} - v^{\tilde{a}} \right) u^q \right).$$

□

**Corollary 69.** *For  $n \in \mathbb{N}, n \geq 3$ , and all integers  $1 \leq a \leq b$  the set  $T_{a,b}^{(n)}$  is finite. Moreover the following inclusion is true*

$$T_{a,b}^{(3)} \subset \{(a, b, d_3, \dots, d_n) \in (\mathbb{N}^*)^n : d_3, \dots, d_n < \text{lcm}(a, b) - r\},$$

where  $r$  is defined as in Theorem 68.

*Proof.* If for some  $i \in \{3, \dots, n\}$  we have  $d_i \geq \text{lcm}(a, b) - r$  (actually we can think that  $i = n$ , because of the inequalities  $d_3 \leq \dots \leq d_n$ ) then by Theorem 68, there exists a tame automorphism  $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  such that  $\text{mdeg } F = (a, b, d_i)$ . Now it is enough to use Proposition 15.  $\square$

## 9. MULTIDEGREE OF THE INVERSE OF POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

In [41] Rusek and Winiarski proved that for all automorphisms  $F$  of  $\mathbb{C}^n$  the equality  $\deg F^{-1} \leq (\deg F)^{n-1}$  holds true and hence  $\deg F^{-1} = \deg F$  for  $n = 2$ . Here we give complete information about  $\text{mdeg } F^{-1}$  for  $F \in \text{Aut}(\mathbb{C}^2)$ .

**9.1. Multidegree and the length of the automorphism of  $\mathbb{C}^2$ .** Here we establish the relations between multidegree of a given automorphism of  $\mathbb{C}^2$  and the length of it. We start with the following

**Lemma 70.** *If  $(P, Q) \in \text{Aut}(\mathbb{C}^2)$  is such that  $\deg P < \deg Q$ , then there is a polynomial  $f \in \mathbb{C}[T]$  with  $\deg f > 1$  such that:*

- (1)  $\deg(Q - f(P)) < \deg P$  if  $\deg P > 1$  or
- (2)  $\deg(Q - f(P)) = 1$  if  $\deg P = 1$ .

*Proof.* Since  $\deg Q > \deg P \geq 1$ , we have  $\deg Q + \deg P > 2$  and  $\text{Jac}(\overline{P}, \overline{Q}) = 0$  (because  $\text{Jac}(P, Q) \in \mathbb{C}^*$ ). By Lemma 4,

$$\overline{P} = \alpha h^{n_1} \quad \overline{Q} = \beta h^{n_2}$$

for some  $\alpha, \beta \in \mathbb{C}^*, n_1, n_2 \in \mathbb{N}^*$  and some homogeneous polynomial  $h \in \mathbb{C}[X, Y]$ . Since  $\deg \overline{P} | \deg \overline{Q}$ , we have  $n_1 | n_2$  and so  $\overline{Q} = c_1 \overline{P}^{k_1}$  for some  $c_1 \in \mathbb{C}^*$  and  $k_1 = \frac{n_2}{n_1}$ . Now  $\deg(Q - c_1 P^{k_1}) < \deg Q$ , and if  $\deg(Q - c_1 P^{k_1}) < \deg P$  or  $\deg(Q - c_1 P^{k_1}) = \deg P = 1$ , then we are done. And, if  $\deg(Q - c_1 P^{k_1}) > \deg P$  or  $\deg(Q - c_1 P^{k_1}) = \deg P > 1$ , then we can repeat the above arguments for  $\overline{Q - c_1 P^{k_1}}$  and  $\overline{P}$  to obtain  $c_2 \in \mathbb{C}^*$  and  $k_2 < k_1$  such that  $\overline{Q - c_1 P^{k_1}} = c_2 \overline{P}^{k_2}$ . Then,

$$\deg(Q - c_1 P^{k_1} - c_2 P^{k_2}) < \deg(Q - c_1 P^{k_1})$$

and we can proceed inductively.  $\square$

Now we can prove the following

**Proposition 71.** *If  $F \in \text{Aut}(\mathbb{C}^2)$ , then there is a number  $l \in \mathbb{N}$  (including zero), affine automorphisms  $L_1, L_2$  of  $\mathbb{C}^2$  and triangular automorphisms  $T_1, \dots, T_l$  of the forms*

$$(50) \quad T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2 \quad \text{for } i = 1, 3, \dots$$

$$(51) \quad T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2 \quad \text{for } i = 2, 4, \dots$$

with  $\deg f_i > 1$ , such that

$$F = L_2 \circ T_l \circ \dots \circ T_1 \circ L_1.$$

Moreover, the number  $l$  is unique, and one can require that  $T_i$ ,  $i = 1, \dots, l$  are of the form (50) for even  $i$  and of the form (51) for odd  $i$ .

*Proof.* Let  $F = (F_1, F_2)$ . If  $\deg F_1 = \deg F_2 = 1$ , then  $F$  is an affine mapping and we have  $F = L_2 \circ L_1$  for  $L_2 = \text{id}_{\mathbb{C}^2}$  and  $L_1 = F$ .

If  $\deg F_1 = \deg F_2 > 1$ , then  $\text{Jac}(\overline{F_1}, \overline{F_2}) = 0$  (because  $\text{Jac}(F_1, F_2) \in \mathbb{C}^*$ ). Thus, by Lemma 4

$$\overline{F_1} = \alpha h^n \quad \overline{F_2} = \beta h^n$$

for some  $\alpha, \beta \in \mathbb{C}^*$ ,  $n \in \mathbb{N}^*$  and some homogeneous polynomial  $h \in \mathbb{C}[X, Y]$ . Let  $L_2(x, y) = \left(x + \frac{\alpha}{\beta}y, y\right)$  and

$$(G_1, G_2) = L_2^{-1} \circ F.$$

Then  $\deg G_2 = \deg F_2$  (actually  $G_2 = F_2$ ) and  $\deg G_1 < \deg G_2$ . Hence we can assume that  $\deg F_1 \neq \deg F_2$ , and without loss of generality that  $\deg F_1 < \deg F_2$  (if  $\deg F_1 > \deg F_2$ , then for  $(G_1, G_2) = L_2^{-1} \circ F$ , where  $L_2(x, y) = (y, x)$ , we have  $\deg G_1 < \deg G_2$ ).

By Lemma 70, we obtain a polynomial  $f \in \mathbb{C}[T]$ ,  $\deg f > 1$ , such that for  $T_1(x, y) = (x, y + f(x))$  and  $(G_1, G_2) = T_1^{-1} \circ F$  we have  $\deg G_2 < \deg G_1$  or  $\deg G_2 = \deg G_1 = 1$ . In the second case  $(G_1, G_2)$  is an affine map and for  $L_1 = (G_1, G_2)$  we have  $F = T_1 \circ L_1$ , so we are done. And in the first case we can next time use Lemma 70 and proceed inductively.

Thus we can assume that  $F = \tilde{L}_2 \circ \tilde{T}_1 \circ \dots \circ \tilde{T}_l \circ \tilde{L}_1$ , where  $\tilde{L}_1, \tilde{L}_2 \in \text{Aff}(\mathbb{C}^2)$  and  $\tilde{T}_i$  are of the forms (50), (51). Let us set

$$T_i = \begin{cases} \tilde{T}_{l+1-i}, & \text{for odd } l, \\ L \circ \tilde{T}_{l+1-i} \circ L, & \text{for even } l, \end{cases}$$

$$L_1 = \begin{cases} \tilde{L}_1, & \text{for odd } l, \\ L \circ \tilde{L}_1, & \text{for even } l, \end{cases} \quad L_2 = \begin{cases} \tilde{L}_2, & \text{for odd } l, \\ \tilde{L}_2 \circ L, & \text{for even } l, \end{cases}$$

where  $L(x, y) = (y, x)$ . Then one can check that  $F = L_2 \circ T_l \circ \dots \circ T_1 \circ L_1$ .

To see that  $l$  is unique it is enough to notice that  $L \circ T_j \circ L \in J(\mathbb{C}^2) \setminus \text{Aff}(\mathbb{C}^2)$ ,  $j = 1, 3, \dots$  and  $T_j \in J(\mathbb{C}^2) \setminus \text{Aff}(\mathbb{C}^2)$ ,  $j = 2, 4, \dots$ , and so

$$F = \hat{L}_2 \circ \dots \circ L \circ (L \circ T_3 \circ L) \circ L \circ T_2 \circ L \circ (L \circ T_1 \circ L) \circ (L \circ L_1),$$

is the amalgamated representation of  $F$  for a suitable chosen sets  $\Phi$  and  $\Psi$  (see Definition and Proposition), where

$$\hat{L}_2 = \begin{cases} \tilde{L}_2, & \text{for even } l, \\ \tilde{L}_2 \circ L, & \text{for odd } l. \end{cases}$$

To see that the last statement holds true, one can write

$$F = (L_2 \circ L) \circ (L \circ T_l \circ L) \circ \dots \circ (L \circ T_1 \circ L) \circ (L \circ L_1).$$

□

**Definition 6.** Let  $F \in \text{Aut}(\mathbb{C}^2)$  be a polynomial automorphism. The number  $l$  from Proposition 71 is called length of  $F$  and denoted  $\text{length } F$ .

In what follows we will use the following numerical object.

**Definition 7.** Let  $k \in \mathbb{N}^*$  and let  $k = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$  be its prime decomposition. Then by  $l(k)$  we denote the number  $\alpha_1 + \cdots + \alpha_r$ .

Obviously, we have  $l(k_1 k_2) = l(k_1) + l(k_2)$ , for all  $k_1, k_2 \in \mathbb{N}^*$ , and  $l(k) \geq 1$  for  $k > 1$ .

**Theorem 72.** Let  $F \in \text{Aut}(\mathbb{C}^2)$ . Then:

- (1) if  $\text{length } F = 1$ , then  $\text{mdeg } F \in \{(1, d), (d, 1), (d, d)\}$ , where  $1 < d$ ,
- (2) if  $\text{length } F = 2$ , then either  $\text{mdeg } F \in \{(d_1, d_2), (d_2, d_1)\}$  with  $1 < d_1 < d_2, d_1 | d_2$  or  $\text{mdeg } F = (d, d)$  with  $l(d) \geq 2$  (in particular  $d > 1$  is a composite number),
- (3) if  $\text{length } F \geq 3$ , then either  $\text{mdeg } F \in \{(d_1, d_2), (d_2, d_1)\}$  with  $1 < d_1 < d_2, d_1 | d_2, l(d_1) \geq \text{length } F - 1$  or  $\text{mdeg } F = (d, d)$  with  $l(d) \geq \text{length } F$ .

*Proof.* (1) Since  $\text{length } F = 1$ ,  $F = L_2 \circ T \circ L_1$ , where  $L_1, L_2 \in \text{Aff}(\mathbb{C}^2)$  and  $T$  is of the form  $T : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f(x)) \in \mathbb{C}^2$ , with  $\deg f > 1$ . Thus  $\text{mdeg}(T \circ L_1) = (1, d)$ , where  $d = \deg f$ , and then one can easily check that  $\text{mdeg}(L_2 \circ T \circ L_1) \in \{(1, d), (d, 1), (d, d)\}$ .

(2) Since  $\text{length } F = 2$ ,  $F = L_2 \circ T_2 \circ T_1 \circ L_1$ , where  $L_1, L_2 \in \text{Aff}(\mathbb{C}^2)$  and  $T_1, T_2$  are of the following forms

$$\begin{aligned} T_1 &: \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_1(x)) \in \mathbb{C}^2, \\ T_2 &: \mathbb{C}^2 \ni (x, y) \mapsto (x + f_2(y), y) \in \mathbb{C}^2, \end{aligned}$$

with  $\deg f_1, \deg f_2 > 1$ . Thus  $\text{mdeg}(T_1 \circ L_1) = (1, \deg f_1)$ , and then  $\text{mdeg}(T_2 \circ T_1 \circ L_1) = (d_2, d_1)$ , where  $d_1 = \deg f_1, d_2 = \deg f_2 \cdot \deg f_1$ . Since  $\deg f_1, \deg f_2 > 1$ , it follows that  $l(d_2) = l(\deg f_1) + l(\deg f_2) \geq 2$ . Now, one can easily see that  $\text{mdeg}(L_2 \circ T_2 \circ T_1 \circ L_1) \in \{(d_1, d_2), (d_2, d_1), (d_2, d_2)\}$ .

(3) Since  $l = \text{length } F \geq 3$ ,  $F = L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1$ , where  $L_1, L_2 \in \text{Aff}(\mathbb{C}^2)$  and  $T_1, \dots, T_l$  are of the following forms

$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2,$$

for even  $i$ , and

$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2,$$

for odd  $i$ , with  $\deg f_i > 1$  for  $i = 1, \dots, l$ . Now, one can easily check that

$$\text{mdeg}(T_l \circ \cdots \circ T_1 \circ L_1) = \begin{cases} \left( \prod_{j=1}^l \deg f_j, \prod_{j=1}^{l-1} \deg f_j \right), & \text{for even } l, \\ \left( \prod_{j=1}^{l-1} \deg f_j, \prod_{j=1}^l \deg f_j \right), & \text{for odd } l. \end{cases}$$

Let

$$d_2 = \prod_{j=1}^l \deg f_j \quad \text{and} \quad d_1 = \prod_{j=1}^{l-1} \deg f_j.$$

Then,  $\text{mdeg}(T_l \circ \cdots \circ T_1 \circ L_1) = (d_1, d_2)$  for odd  $l$ , and  $\text{mdeg}(T_l \circ \cdots \circ T_1 \circ L_1) = (d_2, d_1)$  for even  $l$ .

Since  $\deg f_i > 1$ , for  $i = 1, \dots, l$ , we have

$$l(d_1) \geq l(\deg f_1) + \cdots + l(\deg f_{l-1}) \geq l - 1$$

and

$$l(d_2) \geq l(\deg f_1) + \cdots + l(\deg f_l) \geq l.$$

Of course, as in the previous case, we have

$$\text{mdeg}(L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1) \in \{(d_1, d_2), (d_2, d_1), (d_2, d_2)\}.$$

□

**Theorem 73.** *Let  $F \in \text{Aut}(\mathbb{C}^2)$  be arbitrary polynomial automorphism with  $\text{mdeg } F = (d_1, d_2)$ ,  $d_1 \leq d_2$ . Then  $\text{length } F \leq \min\{l(d_2), l(d_1) + 1\}$ .*

*Proof.* This is a consequence of Theorem 72.  $\square$

**9.2. The case of length 1.** Here we consider the situation with  $\text{length } F = 1$ . Because of Theorem 72, this simple situation is described by the following result.

**Theorem 74.** *Let  $F \in \text{Aut}(\mathbb{C}^2)$ ,  $\text{length } F = 1$  and  $\text{mdeg } F \in \{(1, d), (d, d)\}$ , with  $1 < d$ . Then*

$$\text{mdeg } F^{-1} \in \{(1, d), (d, 1), (d, d)\}.$$

*Proof.* Since  $\text{length } F = 1$ ,  $F = L_2 \circ T \circ L_1$ , where  $T$  is a triangular automorphism of the form  $T : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f(x)) \in \mathbb{C}^2$ , with  $\deg f > 1$ , and  $L_1, L_2 \in \text{Aff}(\mathbb{C}^2)$ . Let us notice that  $\deg f = \deg T = \deg F = d$ . Thus  $\text{mdeg}(T^{-1} \circ L_2^{-1}) = (1, d)$ . Now, it is easy to see that

$$\text{mdeg } F^{-1} = \text{mdeg}(L_1^{-1} \circ T^{-1} \circ L_2^{-1}) \in \{(1, d), (d, 1), (d, d)\}.$$

$\square$

The following two examples show that all possibilities described in the above theorem are realized.

**Example 4.** *Let  $d \in \mathbb{N} \setminus \{0, 1\}$ . Let us put*

$$F_a = T, \quad F_b = T \circ L_b, \quad \text{and} \quad F_c = T \circ L_c,$$

where  $T(x, y) = (x, y + x^d)$ ,  $L_b(x, y) = (y, x)$  and  $L_c(x, y) = (x + y, y)$ . One can check that

$$\text{mdeg } F_a = \text{mdeg } F_b = \text{mdeg } F_c = (1, d)$$

and

$$\text{mdeg } F_a^{-1} = (1, d), \quad \text{mdeg } F_b^{-1} = (d, 1), \quad \text{mdeg } F_c^{-1} = (d, d).$$

**Example 5.** *Let  $d \in \mathbb{N} \setminus \{0, 1\}$  and put*

$$F_a = L_c \circ T, \quad F_b = L_c \circ T \circ L_b, \quad \text{and} \quad F_c = L_c \circ T \circ L_c,$$

where  $T, L_b$  and  $L_c$  are defined as in the previous example. One can check that

$$\text{mdeg } F_a = \text{mdeg } F_b = \text{mdeg } F_c = (d, d)$$

and

$$\text{mdeg } F_a^{-1} = (1, d), \quad \text{mdeg } F_b^{-1} = (d, 1), \quad \text{mdeg } F_c^{-1} = (d, d).$$

**9.3. The case  $(d_1, d_2)$ .** Here we investigate the situation with  $\text{mdeg } F = (d_1, d_2)$ ,  $d_1 \neq d_2$  and  $\text{length } F > 1$ . Of course, without loss of generality, we can assume that  $d_1 < d_2$ . Because of Theorem 72, the situation is described by the following two theorems.

**Theorem 75.** *Let  $F \in \text{Aut}(\mathbb{C}^2)$ ,  $\text{length } F = 2$  and  $\text{mdeg } F = (d_1, d_2)$ , with  $1 < d_1 < d_2$ ,  $d_1 | d_2$ . Then*

$$\text{mdeg } F^{-1} \in \left\{ \left( d_2, \frac{d_2}{d_1} \right), \left( \frac{d_2}{d_1}, d_2 \right), (d_2, d_2) \right\}.$$

*Proof.* Since  $\text{length } F = 2$ ,  $F = L_2 \circ T_2 \circ T_1 \circ L_1$ , where  $T_1, T_2$  are triangular (and not affine) automorphisms and  $L_1, L_2 \in \text{Aff}(\mathbb{C}^2)$ . We can assume that  $T_1$  and  $T_2$  are of the following form:

$$\begin{aligned} T_1 & : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_1(y), y) \in \mathbb{C}^2, \\ T_2 & : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_2(x)) \in \mathbb{C}^2. \end{aligned}$$

Then,  $\text{mdeg}(T_1 \circ L_1) = (\deg f_1, 1)$  and  $\text{mdeg}(T_2 \circ T_1 \circ L_1) = (\deg f_1, \deg f_2 \cdot \deg f_1)$ . Thus, we have  $\deg f_1 = d_1$  and  $\deg f_2 = \frac{d_2}{d_1}$ . Now one can easily check that

$$\text{mdeg}(T_2^{-1} \circ L_2^{-1}) = (1, \deg f_2) = \left(1, \frac{d_2}{d_1}\right)$$

and

$$\text{mdeg}(T_1^{-1} \circ T_2^{-1} \circ L_2^{-1}) = (\deg f_2 \cdot \deg f_1, \deg f_2) = \left(d_2, \frac{d_2}{d_1}\right).$$

Since  $F^{-1} = L_1^{-1} \circ T_1^{-1} \circ T_2^{-1} \circ L_2^{-1}$ , the result follows.  $\square$

The following example shows that all possibilities described in the above theorem are realized.

**Example 6.** Let  $d_1, d_2 \in \mathbb{N}$  be such that  $1 < d_1 < d_2, d_1 | d_2$ . Put

$$\begin{aligned} T_1 & : \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{d_1}, y) \in \mathbb{C}^2, \\ T_2 & : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^\delta) \in \mathbb{C}^2, \end{aligned}$$

where  $\delta = \frac{d_2}{d_1}$ , and

$$F_a = T_2 \circ T_1, \quad F_b = T_2 \circ T_1 \circ L_b, \quad F_c = T_2 \circ T_1 \circ L_c,$$

where  $L_b(x, y) = (y, x)$  and  $L_c(x, y) = (x, y + x)$ . One can check that

$$\text{mdeg } F_a = \text{mdeg } F_b = \text{mdeg } F_c = (d_1, d_2)$$

and

$$\text{mdeg } F_a^{-1} = \left(d_2, \frac{d_2}{d_1}\right), \quad \text{mdeg } F_b^{-1} = \left(\frac{d_2}{d_1}, d_2\right), \quad \text{mdeg } F_c^{-1} = (d_2, d_2).$$

**Theorem 76.** Let  $F \in \text{Aut}(\mathbb{C}^2)$ ,  $\text{length } F \geq 3$  and  $\text{mdeg } F = (d_1, d_2)$ , with  $1 < d_1 < d_2, d_1 | d_2$ . Then

$$\text{mdeg } F^{-1} \in \left\{ \left(d_2, \frac{d_2}{a}\right), \left(\frac{d_2}{a}, d_2\right), (d_2, d_2) : a \in \mathcal{A}_F \right\},$$

where  $\mathcal{A}_F = \{a : 1 < a < d_1, a | d_1, l(\frac{d_1}{a}) \geq \text{length } F - 2\}$ .

*Proof.* Let  $l = \text{length } F$ . Then  $F$  can be written in the following form

$$F = L_2 \circ T_l \circ \cdots \circ T_1 \circ L_1,$$

where  $T_1, \dots, T_l$  are triangular (and not affine) automorphisms and  $L_1, L_2 \in \text{Aff}(\mathbb{C}^2)$ . We can assume that  $T_i$  are of the following forms

$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x + f_i(y), y) \in \mathbb{C}^2$$

for odd  $i$ , and

$$T_i : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + f_i(x)) \in \mathbb{C}^2$$



for even  $i$ . Now, one can check that:

$$\text{mdeg}(T_l \circ \dots \circ T_1 \circ L_1) = \begin{cases} \left( \prod_{j=1}^l \deg f_j, \prod_{j=1}^{l-1} \deg f_j \right), & \text{for odd } l, \\ \left( \prod_{j=1}^{l-1} \deg f_j, \prod_{j=1}^l \deg f_j \right), & \text{for even } l. \end{cases}$$

In both cases we have

$$\prod_{j=1}^l \deg f_j = d_2 \quad \text{and} \quad \prod_{j=1}^{l-1} \deg f_j = d_1.$$

Let  $a = \deg f_1$ . Since  $T_i$  are not affine,  $\deg f_i > 1$ . Since also  $l \geq 3$  (in other words,  $l-1 > 1$ ),  $a$  is a proper divisor of  $d_1$  and  $l \left( \frac{d_1}{a} \right) = l(\deg f_2 \cdots \deg f_{l-1}) \geq l-2$ .

Now, one can check that

$$\text{mdeg}(T_1^{-1} \circ \dots \circ T_l^{-1} \circ L_2^{-1}) = \left( \prod_{j=1}^l \deg f_j, \prod_{j=2}^l \deg f_j \right) = \left( d_2, \frac{d_2}{a} \right).$$

Since  $F^{-1} = L_1^{-1} \circ T_1^{-1} \circ \dots \circ T_l^{-1} \circ L_2^{-1}$ , the result follows.  $\square$

Also in this case all possibilities are realized, as the following example shows.

**Example 7.** Let  $d_1, d_2 \in \mathbb{N}$  be such that  $1 < d_1 < d_2$ ,  $d_1 | d_2$ , and let  $l \leq l(d_1) + 1$  be an even number. Assume also that  $a$  is a proper divisor of  $d_1$  such that  $l \left( \frac{d_1}{a} \right) \geq l-2$ . Take positive integers  $a_2, \dots, a_{l-1}$  such that

$$d_1 = a \cdot a_2 \cdots a_{l-1}.$$

Such integers exist, because  $l \left( \frac{d_1}{a} \right) \geq l-2$ . Now put:

$$\begin{aligned} T_1 &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^a, y) \in \mathbb{C}^2, \\ T_2 &: \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^{a_2}) \in \mathbb{C}^2, \\ T_3 &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{a_3}, y) \in \mathbb{C}^2, \\ &\vdots \\ T_{l-1} &: \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{a_{l-1}}, y) \in \mathbb{C}^2, \\ T_l &: \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^\delta) \in \mathbb{C}^2, \end{aligned}$$

where  $\delta = \frac{d_2}{d_1}$ . Let us, also, set:

$$F_a = T_l \circ \dots \circ T_1, \quad F_b = T_l \circ \dots \circ T_1 \circ L_b,$$

and

$$F_c = T_l \circ \dots \circ T_1 \circ L_c,$$

where  $L_b$  and  $L_c$  are defined as in the previous example. One can check that

$$\text{mdeg } F_a = \text{mdeg } F_b = \text{mdeg } F_c = (d_1, d_2)$$

and

$$\text{length } F = l.$$

It is also easy to see that:

$$\text{mdeg } F_a^{-1} = \left( d_2, \frac{d_2}{a} \right), \quad \text{mdeg } F_b^{-1} = \left( \frac{d_2}{a}, d_2 \right), \quad \text{mdeg } F_c^{-1} = (d_2, d_2).$$

In a similar way one can obtain an example when  $l$  is odd.

The following example shows an application of Theorem 76.

**Example 8.** Let  $F \in \text{Aut}(\mathbb{C}^2)$  be such that  $\text{mdeg } F = (60, 120)$ . Since  $l(60) = l(2^2 \cdot 3 \cdot 5) = 4$ , then  $\text{length } F \leq 5$ .

If  $\text{length } F = 3$ , then

$$\mathcal{A}_F = \{2, 3, 5, 4, 6, 10, 15, 12, 20, 30\},$$

and so, by Theorem 76,

$$\begin{aligned} \text{mdeg } F^{-1} \in & \{(120, 60), (120, 40), (120, 24), (120, 30), (120, 20), \\ & (120, 12), (120, 8), (120, 10), (120, 6), (120, 4), (60, 120), \\ & (40, 120), (24, 120), (30, 120), (20, 120), (12, 120), \\ & (8, 120), (10, 120), (6, 120), (4, 120), (120, 120)\}. \end{aligned}$$

If  $\text{length } F = 4$ , then

$$\mathcal{A}_F = \{2, 3, 5, 4, 6, 10, 15\},$$

and so, by Theorem 76,

$$\begin{aligned} \text{mdeg } F^{-1} \in & \{(120, 60), (120, 40), (120, 24), (120, 30), (120, 20), \\ & (120, 12), (120, 8), (60, 120), (40, 120), (24, 120), \\ & (30, 120), (20, 120), (12, 120), (8, 120), (120, 120)\}. \end{aligned}$$

If  $\text{length } F = 5$ , then

$$\mathcal{A}_F = \{2, 3, 5\},$$

and so, by Theorem 76,

$$\begin{aligned} \text{mdeg } F^{-1} \in & \{(120, 60), (120, 40), (120, 24), \\ & (60, 120), (40, 120), (24, 120), (120, 120)\}. \end{aligned}$$

Moreover, by the previous example, all above listed possibilities are realized.

**9.4. The case  $(d, d)$ .** Using a similar arguments as in the proof of Theorem 76 one can prove the following

**Theorem 77.** Let  $F \in \text{Aut}(\mathbb{C}^2)$ ,  $\text{length } F \geq 2$  and  $\text{mdeg } F = (d, d)$ , with  $1 < d$ . Then:

$$\text{mdeg } F^{-1} \in \left\{ \left( d, \frac{d}{a} \right), \left( \frac{d}{a}, d \right), (d, d) : a \in \mathcal{A}_F \right\},$$

where  $\mathcal{A}_F = \{a : 1 < a < d, a|d, l(\frac{d}{a}) \geq \text{length } F - 1\}$ .

Also in this case all described possibilities are realized, as the following example shows (this example is a modification of the example given after Theorem 76).

**Example 9.** Let  $d \in \mathbb{N}$  and  $l \geq 2$  be a even number such that let  $l \leq l(d)$ . Assume, also, that  $a$  is a proper divisor of  $d$  such that  $l(\frac{d}{a}) \geq l - 1$ . Take positive integers  $a_2, \dots, a_l$  such that

$$d = a \cdot a_2 \cdots a_l.$$

Such integers exist, because  $l(\frac{d}{a}) \geq l - 1$ . Let  $T_1, \dots, T_{l-1}$  be defined as in Example 7 and put

$$T_l : \mathbb{C}^2 \ni (x, y) \mapsto (x, y + x^{a_l}) \in \mathbb{C}^2.$$

Let us also set:

$$F_a = L \circ T_l \circ \dots \circ T_1, \quad F_b = L \circ T_l \circ \dots \circ T_1 \circ L_b,$$

and

$$F_c = L \circ T_l \circ \dots \circ T_1 \circ L_c,$$

where  $L_b(x, y) = (y, x)$ ,  $L_c(x, y) = (x, y + x)$  and  $L(x, y) = (x + y, y)$ . Then one can check that

$$\text{mdeg } F_a = \text{mdeg } F_b = \text{mdeg } F_c = (d, d), \quad \text{length } F = l,$$

and

$$\text{mdeg } F_a^{-1} = \left(d, \frac{d}{a}\right), \quad \text{mdeg } F_b^{-1} = \left(\frac{d}{a}, d\right), \quad \text{mdeg } F_c^{-1} = (d, d).$$

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